Journal of Combinatorial Theory, Series A 132 (2015) 58–101

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Circular planar electrical networks: Posets and positivity

Joshua Alman, Carl Lian, Brandon Tran

Department of Mathematics, Massachusetts Institute of Technology, United States

A R T I C L E   I N F O

Article history:
Received 15 January 2014
Available online xxxx

Keywords:
Planar graph
Resistor network
Network response
Poset
Laurent phenomenon
Electrical positroid

A B S T R A C T

Following de Verdière–Gitler–Vertigan and Curtis–Ingerman–Morrow, we prove a host of new results on circular planar electrical networks. We first construct a poset EP_n of electrical networks with n boundary vertices, and prove that it is graded by number of edges of critical representatives. We then answer various enumerative questions related to EP_n, adapting methods of Callan and Stein–Everett. Finally, we study certain positivity phenomena of the response matrices arising from circular planar electrical networks. In doing so, we introduce electrical positroids, extending work of Postnikov, and discuss a naturally arising example of a Laurent phenomenon algebra, as studied by Lam–Pylyavskyy.

1. Introduction

Circular planar electrical networks are objects from classical physics: given a resistor network, one can compute its response to imposed voltages via the *Dirichlet-to-Neumann map*. An inverse boundary problem for electrical networks was studied in detail by de Verdière, Gitler, and Vertigan [5] and Curtis, Ingerman, and Morrow [4]: given the Dirichlet-to-Neumann map, can the network be recovered?

E-mail addresses: jalman@mit.edu (J. Alman), clian@math.mit.edu (C. Lian), btran115@mit.edu (B. Tran).

http://dx.doi.org/10.1016/j.jcta.2014.11.004

In general, the answer is "no," though much can be said about the information that can be recovered. If, for example, the underlying graph of the electrical network is known and is *critical*, the resistances are uniquely determined [4, Theorem 2]. Moreover, any two networks that produce the same response matrix can be related by a certain class of combinatorial transformations, the *local equivalences* [5, Théorème 4].

The goal of this paper is to study more closely the rich theory of circular planar electrical networks. We define a poset EP_n of circular planar graphs, under the operations of contraction and deletion of edges, and investigate its properties. By [4, Theorem 4] and [5, Théorème 3], the space of response matrices for circular planar electrical networks of order n decomposes as a disjoint union of open *cells*, each diffeomorphic to a product of copies of the positive real line. We then have:

Theorem 3.1.3. *If $[H]$ and $[G]$ are equivalence classes of circular planar graphs, then $[H] \leq [G]$ in EP_n if and only if $\Omega(H) \subset \overline{\Omega(G)}$, where $\Omega(H)$ denotes the space of response matrices for conductances on H.*

Using the important tool of *medial graphs* developed in [4] and [5], we also prove:

Theorem 3.2.4. *EP_n is graded by number of edges of critical representatives.*

We then obtain the following enumerative properties of EP_n via medial graphs, adapting techniques of Callan [3] and Stein and Everett [21].

Theorem 4.2.5. *Put $X_n = |EP_n|$, the number of equivalence classes of electrical networks of order n. Then:*

(a) $X_1 = 1$ *and*

$$X_n = 2(n-1)X_{n-1} + \sum_{j=2}^{n-2}(j-1)X_jX_{n-j}.$$

(b) $[t^{n-1}]X(t)^n = n \cdot (2n-3)!!$, *where $X(t)$ is the generating function for the sequence* $\{X_i\}$.

(c) $X_n/(2n-1)!! \to e^{-1/2}$ *as* $n \to \infty$.

Associated to any circular planar electrical network of order n is its $n \times n$ *response matrix*, the Dirchlet-to-Neumann map expressed in a canonical basis. Response matrices are characterized in [4, Theorem 4] as the symmetric matrices with row sums equal to zero and *circular minors* non-negative. Furthermore, the strictly positive minors can be identified combinatorially using [4, Lemma 4.2].

A natural question that arises is: which sets of circular minors can be positive, while the others are zero? Postnikov [16] studied a similar question in the *totally nonnegative*

Grassmannian: for $k \times n$ matrices A, with $k < n$ and all $k \times k$ minors nonnegative, which sets (matroids) of $k \times k$ minors can be the set of positive minors of A[1]? These special matroids, called *positroids* by Knutson, Lam, and Speyer [11], were found in [16] to index many interesting combinatorial objects. Two of these objects, plabic graphs and alternating strand diagrams, are highly similar to circular planar electrical networks and medial graphs.

In light of this question, we give an axiomatization of *electrical positroids*, motivated by the Grassmann–Plücker relations. The following theorem shows that this notion is, in a sense, a natural extension of Postnikov's theory of positroids:

Theorem 5.2.1. *A set S of circular pairs is the set of positive circular minors of a response matrix if and only if S is an electrical positroid.*

Finally, we consider *positivity tests* for response matrices. In [8], Fomin and Zelevinsky describe various positivity tests for *totally positive matrices*: given an $n \times n$ matrix, there exist sets of n^2 minors whose positivity implies the positivity of all minors. Moreover, positivity tests are related to each other combinatorially via *double wiring diagrams*. Fomin and Zelevinsky later introduced *cluster algebras* in [9], in part, to study similar positivity phenomena.

In a similar way, we describe positivity tests of size $\binom{n}{2}$ for $n \times n$ matrices. Some such sets were first described by Kenyon [10, §4.5.3]. While they do not form clusters in a cluster algebra, our positivity tests form clusters in a *Laurent phenomenon algebra*, as introduced by Lam and Pylyavskyy in [12]. We find:

Theorem 6.2.16. *There exists an LP algebra $\mathcal{LM}_n$, isomorphic to the polynomial ring on $\binom{n}{2}$ generators, with an initial seed $\mathcal{D}_n$ of diametric circular minors. $\mathcal{D}_n$ is a positivity test for circular minors, and furthermore, all "Plücker clusters" in $\mathcal{LM}_n$, that is, clusters of circular minors, are positivity tests.*

Moreover, $\mathcal{LM}_n$ is "doubly-covered" by a cluster algebra $\mathcal{CM}_n$ that behaves similarly to $\mathcal{LM}_n$ when we restrict to certain types of mutations. Further investigation of the clusters leads to an analogue of weak separation, as studied by Oh, Speyer, and Postnikov [15] and Scott [19]. Conjecturally, the "Plücker clusters," of $\mathcal{LM}_n$ correspond exactly to the maximal pairwise weakly separated sets of circular pairs. We conjecture that these maximal pairwise weakly separated sets are related to each other by mutations corresponding to the Grassmann–Plücker relations, and present evidence to this end.

The roadmap of the paper is as follows. We briefly review terminology and known results in Section 2, where we also establish some basic properties of electrical networks. In Section 3, we define the poset EP_n and establish its most important properties in

[1] It is worth noting that the introduction of the totally nonnegative Grassmannian was motivated in part by the study of electrical networks, see [16, p. 2].

Theorems 3.1.3 and 3.2.4. The study of enumerative properties of EP_n is undertaken in Section 4, where we prove the three parts of Theorem 4.2.5. In Section 5, we motivate and introduce electrical positroids, the main result being Theorem 5.2.1. Finally, in Section 6, we construct $\mathcal{LM}_n$ using positivity tests and prove Theorem 6.2.16, then conclude by establishing weak forms of Conjecture 6.3.4, which relates the clusters of $\mathcal{LM}_n$ to positivity tests and our analogue of weak separation.

2. Electrical networks

We begin a systematic discussion of electrical networks by recalling various notions and results from [4]. We will also introduce some new terminology and conventions which will aid our exposition, in some cases deviating from [4].

2.1. Circular planar electrical networks, up to equivalence

Definition 2.1.1. A **circular planar electrical network** is a circular planar graph (i.e., a planar graph embedded in a disk, where vertices on the boundary of the disk are referred to as *boundary vertices*) Γ, together with a **conductance map** $\gamma : E(\Gamma) \to \mathbb{R}_{>0}$.

To avoid cumbersome language, we will henceforth refer to these objects as **electrical networks**. We will also call the number of boundary vertices of an electrical network (or a circular planar graph) its **order**. We also adopt the following convention: current going in to the disk is measured to be negative. This convention is the opposite of that used in [4], but we will prefer it for the ensuing elegance of the statement of Theorem 2.2.6.

Associated to an electrical network (Γ, γ) is its **response matrix** (see [4, §3]), measuring the network's response to potentials applied at boundary vertices. Two electrical networks $(\Gamma_1, \gamma_1), (\Gamma_2, \gamma_2)$ are **equivalent** if they have the same response matrix. The resulting equivalence relation $\sim$ may be described combinatorially:

Theorem 2.1.2. *(See [5, Théorème 4].) Two electrical networks are equivalent if and only if they are related by a sequence of* ***local equivalences*** *(and their inverses): removal of self-loops or spikes, replacement of edges in series or in parallel, or Y-Δ transformations.*

The relation $\sim$ is also an equivalence relation on underlying circular planar graphs, and, when there is no likelihood for confusion, we will often mean this underlying graph when referring to an "electrical network."

2.2. Circular pairs and circular minors

Circular pairs and circular minors are central to the characterization of response matrices.

Definition 2.2.1. Let $P = \{p_1, p_2, \dots, p_k\}$ and $Q = \{q_1, q_2, \dots, q_k\}$ be disjoint ordered subsets of the boundary vertices of an electrical network (Γ, γ). We say that $(P; Q)$ is a **circular pair** if $p_1, \dots, p_k, q_k, \dots, q_1$ are in clockwise order around the circle. We will refer to k as the **size** of the circular pair.

Remark 2.2.2. We will take $(P; Q)$ to be the same circular pair as $(\widetilde{Q}; \widetilde{P})$, where $\widetilde{P}$ denotes the ordered set P with its elements reversed.

Definition 2.2.3. Let $(P; Q)$ and (Γ, γ) be as in Definition 2.2.1. We say that there is a **connection** from P to Q in Γ if there exists a collection of vertex-disjoint paths from p_i to q_i in Γ, and furthermore each path in the collection contains no boundary vertices other than its endpoints. We denote the set of circular pairs $(P; Q)$ for which P is connected to Q by $\pi(\Gamma)$.

Definition 2.2.4. Let $(P; Q)$ and (Γ, γ) be as in Definition 2.2.1, and let M be the response matrix. We define the **circular minor** associated to $(P; Q)$ to be the determinant of the $k \times k$ matrix $M(P; Q)$ with $M(P; Q)_{i,j} = M_{p_i, q_j}$.

Remark 2.2.5. When there is no ambiguity, we refer to submatrices and their determinants both as minors, interchangeably.

We are interested in circular minors and connections because of the following results, which are immediate corollaries of [4, Theorem 4] and [4, Theorem 4.2]:

Theorem 2.2.6. *Let M be an $n \times n$ matrix. Then:*

(a) *M is the response matrix for an electrical network (Γ, γ) if and only if M is symmetric with row and column sums equal zero, and each of the circular minors $M(P; Q)$ is non-negative.*
(b) *If M is the response matrix for an electrical network (Γ, γ), the positive circular minors $M(P; Q)$ are exactly those for which there is a connection from P to Q.*

2.3. Critical graphs

In this section, we introduce critical graphs, a particular class of circular planar graphs, and give some important properties.

Definition 2.3.1. Let G be a circular planar graph. G is said to be **critical** if, for any removal of an edge via deletion or contraction (note that an edge between two boundary vertices cannot be contracted), there exists a circular pair $(P; Q)$ for which P is connected to Q through G before the edge removal, but not afterward.

Theorem 2.3.2. *(See [5, Théorème 2].) Every equivalence class of circular planar graphs has a critical representative.*

Theorem 2.3.3. *(See [4, Theorem 1].) Suppose G_1, G_2 are critical. Then, G_1 and G_2 are Y-Δ equivalent (that is, related by a sequence of Y-Δ transformations) if and only if $\pi(G_1) = \pi(G_2)$.*

Corollary 2.3.4. *Let G_1, G_2 be arbitrary circular planar graphs. Then, $G_1 \sim G_2$ if and only if $\pi(G_1) = \pi(G_2)$.*

Theorem 2.3.5. *(See [4, Theorem 4].) Suppose that G is critical and has N edges. Put $\pi = \pi(G)$, and let $\Omega(\pi)$ denote the set of response matrices whose positive minors are exactly those corresponding to the elements of π. Then, the map $r_G : \mathbb{R}^N_{>0} \to \Omega(\pi)$, taking the conductances on the edges of G to the resulting response matrix, is a diffeomorphism.*

It follows that the space of response matrices for electrical networks of order n is the disjoint union of the cells $\Omega(\pi)$, some of which are empty. The non-empty cells $\Omega(\pi)$ are those which correspond to critical graphs G with $\pi(G) = \pi$. We will describe how these cells are attached to each other in Proposition 3.1.2. Later, we will prefer to index these cells by their underlying (equivalence classes of) circular planar graphs: $\Omega(G)$ will denote the space of response matrices for conductances on G.

Theorem 2.3.6. *Let (Γ, γ) be an electrical network. The following are equivalent:*

(1) *Γ is critical.*
(2) *Given the response matrix M of (Γ, γ), γ can uniquely be recovered from M and Γ.*
(3) *Γ has the minimal number of edges among elements of its equivalence class.*
(4) *The **medial graph** (see [4, §6]) $\mathcal{M}(\Gamma)$ of Γ is lensless.*

Proof. Apply [4, Lemma 13.1], [4, Lemma 13.2], and Proposition 2.3.4. □

2.4. Medial graphs

Medial graphs are constructed in [4, §6] as a dual object to electrical networks, and will be an important tool in our study thereof. Their significance is already evident from Theorem 2.3.6.

If G is a critical graph, the geodesics of $\mathcal{M}(G)$ consist of n "wires" connecting pairs of the $2n$ boundary medial vertices. Thus, any critical graph G gives a perfect matching of the medial boundary vertices. Furthermore, suppose $H \sim G$ is critical. By Theorem 2.3.3 and Proposition 2.3.4, G and H are related by Y-Δ transformations, so $\mathcal{M}(G)$ and $\mathcal{M}(H)$ are related by motions. In particular, $\mathcal{M}(G)$ and $\mathcal{M}(H)$ match the same pairs of boundary medial vertices, so we have a well-defined map from critical circular planar graph equivalence classes to matchings. In fact, this map is injective:

Proposition 2.4.1. *Suppose that the geodesics of two lensless medial graphs $\mathcal{M}(G), \mathcal{M}(H)$ match the same pairs of medial boundary vertices. Then, the medial $\mathcal{M}(G)$ and $\mathcal{M}(H)$ are related by motions, or equivalently, G and H are Y-Δ equivalent.*

Proof. Implicit in [4, Theorem 7.2]. □

Definition 2.4.2. Given the boundary vertices of a circular planar graph embedded in a disk D, take $2n$ medial boundary vertices as before. A **wiring diagram** is collection of n smooth curves (wires) embedded in D, each of which connects a pair of medial boundary vertices in such a way that each medial boundary vertex has exactly one incident wire. We require that wiring diagrams have no triple crossings or self-loops. As with electrical networks and medial graphs, the **order** of the wiring diagram is defined to be equal to n.

It is immediate from Proposition 2.4.1 that, given a set of boundary vertices, perfect matchings on the set of medial boundary vertices are in bijection with motion-equivalence classes of lensless wiring diagrams. Thus, we have an injection $G \mapsto \mathcal{M}(G)$ from critical graph equivalence classes to motion-equivalence classes of lensless wiring diagram, but this map is not surjective. We describe the image of this injection in the next definition:

Definition 2.4.3. Given boundary vertices $V_1, \ldots, V_n$ and a wiring diagram W on the same boundary circle, a **dividing line** for W is a line V_iV_j with $i \neq j$ such that there does not exist a wire connecting two points on opposite sides of V_iV_j. The wiring diagram is called **full** if it has no dividing lines.

It is obvious that fullness is preserved under motions. Now, suppose that we have a lensless full wiring diagram W; we now define a critical graph $\mathcal{E}(W)$. Let D be the disk in which our wiring diagram is embedded. The wires of W divide D into faces, and it is well-known that these faces can be colored black and white such that neighboring faces have opposite colors.

The condition that W be full means that each face contains at most one boundary vertex. Furthermore, all boundary vertices are contained in faces of the same color; without loss of generality, assume that this color is black. Then place an additional vertex inside each black face which does not contain a boundary vertex. The boundary vertices, in addition to these added interior vertices, form the vertex set for $\mathcal{E}(W)$. Finally, two vertices of $\mathcal{E}(W)$ are connected by an edge if and only if their corresponding faces share a common point on their respective boundaries, which must be an intersection p of two wires of W. This edge is drawn as to pass through p. An example is shown in Fig. 1.

It is straightforward to check that $\mathcal{M}$ and $\mathcal{E}$ are inverse maps. We have thus proven the following result:

Theorem 2.4.4. *The associations $G \mapsto \mathcal{M}(G)$ and $W \mapsto \mathcal{E}(W)$ are inverse bijections between equivalence classes of critical graphs and motion-equivalence classes of full lensless wiring diagrams.*

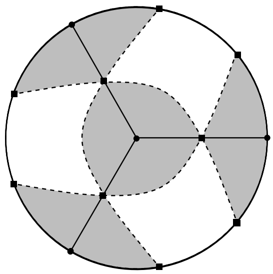


Fig. 1. Recovering an electrical network from its (lensless) medial graph. The square vertices are the vertices of the medial graph, and the dashed edges are edges of the medial graph, while the circle vertices are the vertices of the network. The middle vertex was recovered.

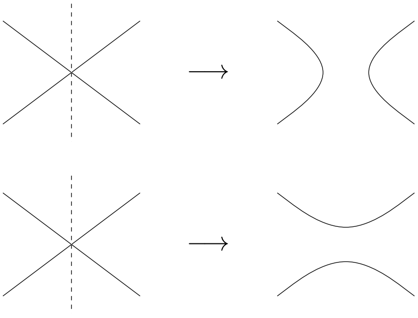


Fig. 2. Breaking a crossing, in two ways.

Finally, let us discuss the analogues of contraction and deletion in medial graphs. Each operation corresponds to the **breaking** of a crossing, as shown in Fig. 2. A crossing may be broken in two ways: breaking outward from the corresponding edge of the underlying electrical network corresponds to contraction, and breaking along the edge corresponds to deletion. In the same way that contraction or deletion of an edge in a critical graph is not guaranteed to yield a critical graph, breaking a crossing in lensless medial graphs does not necessarily yield a lensless medial graph.

Not all breakings of crossings are valid, as some crossings may be broken in a particular way to create a dividing line. In fact, it is straightforward to check that creating a dividing line by breaking a crossing corresponds to contracting a boundary edge, which we also do not allow. Thus, we allow all breakings of crossings as long as no dividing lines are created; such breakings are called **legal**.

3. The electrical poset EP_n

We now consider EP_n, the poset of circular planar graphs under contraction and deletion. We will find that, equivalently, EP_n is the poset of disjoint cells $\Omega(G)$, as defined in Section 2.3 under containment in closure.

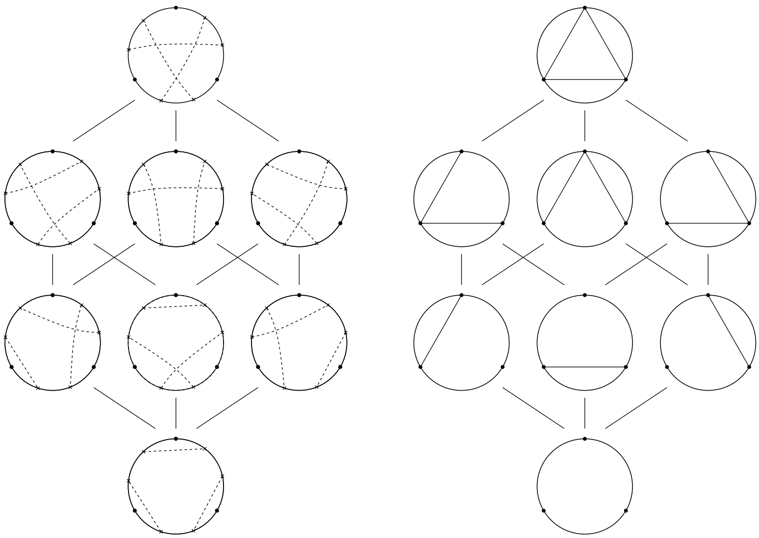


Fig. 3. EP_3.

3.1. Construction

Before constructing EP_n, we need a lemma to guarantee that the order relation will be well-defined.

Lemma 3.1.1. *Let G be a circular planar graph, and suppose that H can be obtained from G by a sequence of contractions and deletions. Consider a circular planar graph G' with $G' \sim G$. Then, there exists a sequence of contractions and deletions starting from G' whose result is some $H' \sim H$.*

Proof. By induction, we may assume that H can be obtained from G by one contraction or one deletion. Furthermore, by Theorem 2.1.2, we may assume by induction that G and G' are related by a local equivalence. From here, the proof is a matter of checking that, for each of the possible *local* equivalences $G \sim G'$ (see Theorem 2.1.2), a sequence of contractions and deletions may be applied to G' to obtain some $H' \sim H$. This is straightforward. □

For distinct equivalence classes $[G], [H]$, we may now define $[H] < [G]$ if, given any $G \in [G]$, there exists a sequence of contractions and deletions that may be applied to G to obtain an element of $[H]$. We thus have a (well-defined) **electrical poset of order** n, denoted EP_n, of equivalence classes of circular planar graphs or order n. If $H \in [H]$ and $G \in [G]$ with $[H] < [G]$, we will write $H < G$.

Fig. 3 shows EP_3, with elements represented as medial graphs (left) and electrical networks (right). Theorem 2.3.2 guarantees that the electrical networks may be taken to be critical. Note that EP_3 is isomorphic to the Boolean Lattice B_3, because all critical graphs of order 3 arise from taking edge-subsets of the top graph.

Let us now give an alternate description of the poset EP_n. Associated to each circular planar graph G, we have an open cell $\Omega(G)$ of response matrices for conductances on G, where $\Omega(G)$ is taken to be a subset of the space Ω_n of symmetric $n \times n$ matrices. It is clear that, if $G \sim G'$, we have, by definition, $\Omega(G) = \Omega(G')$.

Proposition 3.1.2. *Let G be a circular planar graph. Then,*

$$\overline{\Omega(G)} = \bigsqcup_{H \leq G} \Omega(H), \tag{3.1.1}$$

where $\overline{\Omega(G)}$ denotes the closure of $\Omega(G)$ in Ω_n, and the union is taken over equivalence classes of circular planar graphs $H \leq G$ in EP_n.

Because the $\Omega(G)$ are pairwise disjoint when we restrict ourselves to equivalence classes of circular planar graphs (a consequence of Theorems 2.2.6 and 2.3.3), we get:

Theorem 3.1.3. *$[H] \leq [G]$ in EP_n if and only if $\Omega(H) \subset \overline{\Omega(G)}$.*

Proof of Proposition 3.1.2. Without loss of generality, we may take G to be critical. Let N be the number of edges of G. By 2.3.5, the map $r_G : \mathbb{R}^N_{>0} \to \Omega(G) \subset \Omega_n$, sending a collection of conductances of the edges of G the resulting response matrix, is a diffeomorphism. We will describe a procedure for producing a response matrix for any electrical network whose underlying graph H is obtainable from G by a sequence of contractions and deletions (that is, $H \leq G$).

Given $\gamma \in \mathbb{R}^N_{>0}$, write $\gamma = (\gamma_1, \dots, \gamma_N)$. Note that for each $i \in [1, N]$ and fixed conductances $\gamma_1, \dots, \widehat{\gamma_i}, \dots, \gamma_n$, the limit $\lim_{\gamma_i \to 0} r_G(\gamma)$ must exist; indeed, sending the conductance γ_i to zero is equivalent to deleting its associated edge. This fact is most easily seen by physical reasoning: an edge of zero conductance has no current flowing through it, and thus the network may as well not have this edge. Thus, $\lim_{\gamma_i \to 0} r_G(\gamma)$ is just $r_{G'}(\gamma_1, \dots, \widehat{\gamma_i}, \dots, \gamma_n)$, where G' is the result of deleting e from G. Similarly, we find that $\lim_{\gamma_i \to \infty} r_G(\gamma)$ is $r_{G''}(\gamma_1, \dots, \hat{\gamma_i}, \dots, \gamma_n)$, where G'' is the result of contracting e.

It follows easily, then, that for all H which can be obtained from G by a contraction or deletion, we have $\Omega(H) \subset \overline{\Omega(G)}$, because, by the previous paragraph, $\Omega(H) = \mathrm{Im}(r_H) \subset \overline{\Omega(G)}$. By induction, we have the same for all $H \leq G$.

It is left to check that any $M \in \overline{\Omega(G)}$ is in some cell $\Omega(H)$ with $H \leq G$. We have that M is a limit of response matrices $M_1, M_2, \dots \in \Omega(G)$. The determinants of the circular minors of M are limits of determinants of the same minors of the M_i, and thus non-negative. It follows that M is the response matrix for some network H, that is, $M \in \Omega(H)$. We claim that $H \leq G$, which will finish the proof.

Consider the sequence $\{C_k\}$ defined by $C_k = r_G^{-1}(M_k)$, which is a sequence of conductances on G. For each edge $e \in G$, we get a sequence $\{C(e)_k\}$ of conductances of

e in $\{C_k\}$. It is then a consequence of the continuity of r_G, r_G^{-1}, and the existence of the limits $\lim_{\gamma_i \to 0} r_G(\gamma)$, $\lim_{\gamma_i \to \infty} r_G(\gamma)$, that the sequences $\{C(e)_k\}$ each converge to a finite nonnegative limit or otherwise go to $+\infty$.

Furthermore, we claim that for a boundary edge e (that is, one that connects two boundary vertices), $\{C(e)_k\}$ cannot tend to $+\infty$. Suppose, instead, that such is the case, that for some boundary edge $e = V_iV_j$, we have $C(e)_k \to \infty$. Then, note that imposing a positive voltage at V_i and zero voltage at all other boundary vertices sends the current measurement at V_i to $-\infty$ as $C(e)_k \to \infty$. In particular, our sequence $M_1, M_2, \dots$ cannot converge, so we have a contradiction.

To finish, it is clear, for example, using similar ideas to the proof of the first direction, that contracting the edges e for which $C(e)_k \to \infty$ (which can be done because such e cannot be boundary edges) and deleting those for which $C(e)_k \to 0$ yields H. The proof is complete. □

3.2. Gradedness

In this section, we prove our first main theorem, that EP_n is graded.

Proposition 3.2.1. $[G]$ *covers* $[H]$ *in* EP_n *if and only if, for a critical representative* $G \in [G]$*, an edge of* G *may be contracted or deleted to obtain a critical graph in* $[H]$*.*

Proof. First, suppose that G and H are critical graphs such that deleting or contracting an edge of G yields H. Then, if $[G] > [X] > [H]$ for some circular planar graph X, some sequence of at least two deletions or contractions of G yields $H' \sim H$. It is clear that H' has fewer edges than H, contradicting Theorem 2.3.6. It follows that $[G]$ covers $[H]$.

We now proceed to prove the opposite direction. Fix a critical graph G, and let e be an edge of G that can be deleted or contracted in such a way that the resulting graph H is not critical. By way of Lemmas 3.2.2 and 3.2.3, we will first construct $T \sim G$ with certain properties, then, from T, construct a graph G' such that $[G] > [G'] > [H]$. The desired result will then follow: indeed, suppose that $[G]$ covers $[H]$ and $G \in [G]$ is critical. Then, there exists an edge $e \in G$ which may be contracted or deleted to yield $H \in [H]$, and it will also be true that H is critical.

First, we translate to the language of medial graphs. When we break a crossing in the medial graph $\mathcal{M}(G)$, we may create lenses that must be resolved to produce a lensless medial graph. Suppose that our deletion or contraction of $e \in G$ corresponds to breaking the crossing between the geodesics ab and cd in $\mathcal{M}(G)$, where the points a, c, b, d appear in clockwise order on the boundary circle. Let $ab \cap cd = p$, and suppose that when the crossing at p is broken, the resulting geodesics are ad and cb.

For what follows, let $\mathcal{F} = \{f_1, \dots, f_k\}$ denote the set of geodesics f_i in $\mathcal{M}(G)$ such that f_i intersects ab between a and p, and also intersects cd between d and p. In the case that $\mathcal{F}$ is empty, we relabel the points, swapping a with b and c with d. Then define $\mathcal{F}$ as before, with the newly labeled points. Now, if $\mathcal{F}$ is again empty, breaking the crossing at p does not form a lens, so we have a critical graph and are done. Thus, we can assume that $\mathcal{F}$ is nonempty. We now construct T in two steps.

Lemma 3.2.2. *There exists a lensless medial graph K such that:*

- *K is equivalent to $\mathcal{M}(G)$,*
- *geodesics ab and cd still intersect at p, and breaking the crossing at p to give geodesics ad, bc yields a medial graph equivalent to $\mathcal{M}(H)$, and*
- *for $f_i, f_j \in \mathcal{F}$ which cross each other, the crossing $f_i \cap f_j$ lies outside the sector apd.*

Proof. Similar to that of [4, Lemma 6.2]. □

It now suffices to consider the graph K. Let $f_1 \in \mathcal{F}$ be the geodesic intersecting ab at the point v_1 closest to p, and let $w_1 = f_1 \cap cd$.

Lemma 3.2.3. *There exists a lensless medial graph $K' \sim K$, such that:*

- *geodesics ab and cd intersect at p, as before, and breaking the crossing at p to give geodesics ad, bc yields a medial graph equivalent to $\mathcal{M}(H)$, and*
- *no other geodesic of K' enters the triangle with vertices v_1, p, w_1.*

Proof. The lemma follows from a similar argument as that in the proof of Lemma 3.2.2. □

We are now ready to finish the proof of Proposition 3.2.1. Let $T = \mathcal{E}(K')$ (see Theorem 2.4.4). Then, in T, because of the properties of K', contracting e to form the graph $H' \sim H$ forms a pair of parallel edges. Replacing the parallel edges with a single edge gives a circular planar graph H'', which is still equivalent to H. Suppose that e has endpoints B, C and the edges in parallel are formed with A. Then, we have the triangle ABC in T.

Write $S = \pi(T)$ (see Definition 2.2.3) and $S' = \pi(H')$. Because T is critical, $S' \neq S$, so fix $(P; Q) \in S - S'$. Then, it is straightforward to check that any connection $\mathcal{C}$ between P and Q must have used both B and C, but cannot have used the edge BC. Furthermore, $\mathcal{C}$ can use at most one of the edges AB, AC. Indeed, if both AB, AC are used, they appear in the same path γ, but replacing the two edges AB, AC with BC in γ gives a connection between P and Q, but we know that no such connection can use BC, a contradiction. Without loss of generality, suppose that $\mathcal{C}$ does not use AB. Then, deleting AB from T yields a graph G' with $(P; Q) \in G'$, hence G' is not equivalent to H. However, it is clear that deleting BC from G' yields $H'' \sim H$. It follows, then, that in the case in which e is contracted, we have G' such that $[G] > [G'] > [H]$, and hence $[G]$ does not cover $[H]$.

For the case in which we delete $e = ZC$ in T, the argument is similar. □

Theorem 3.2.4. *EP_n is graded by number of edges of critical representatives.*

Proof. First, by Theorem 2.3.6, note that for any $[G] \in EP_n$, all critical representatives of $[G]$ have the same number of edges. Now, we need to show that if $[G]$ covers $[H]$, the number of edges in a critical representative of $[G]$ is one more than the same number for $[H]$. Let $G \in [G]$ be critical. By Proposition 3.2.1, an edge of G may be contracted or deleted to yield a critical representative $H \in [H]$, and it is clear that H has one fewer edge than G. $\square$

Definition 3.2.5. For all non-negative integers r, denote the set of elements of EP_n of rank r by $EP_{n,r}$.

3.3. Toward Eulerianness

In this section, we state various conjectured properties of EP_n, related to Eulerianness. First, we show that EP_n is Eulerian in intervals of length 2.

Lemma 3.3.1. *Suppose $x \in EP_{n,r-1}, z \in EP_{n,r+1}$ with $x < z$. Then, there exist exactly two $y \in EP_{n,r}$ with $x < y < z$.*

Proof. Take x and z to be (equivalence classes of) lensless medial graphs. By Theorem 3.2.4, x may be obtained from z by a sequence of two legal resolutions of crossings. One checks that, in all cases, there are exactly two ways to get from x to z in this way. The details are omitted. $\square$

Conjecture 3.3.2. *EP_n is lexicographically shellable, and hence Cohen–Macaulay, spherical, and Eulerian.*

Refer to [2] for definitions. Indeed, if we have an L-labeling for EP_n, it would follow that the order complex $\Delta(EP_n)$ is shellable and thus Cohen–Macaulay (see [2, Theorem 3.4, Theorem 5.4(C)]). By Lemma 3.3.1, [1, Proposition 4.7.22] would apply, and we would conclude that EP_n is spherical and hence Eulerian.

Using [20], EP_n has been verified to be Eulerian for $n \leq 7$, and the homology of $EP_n - \{\widehat{0}, \widehat{1}\}$ agrees with that of a sphere of the correct dimension, $\binom{n}{2} - 2$, for $n \leq 4$. On the other hand, no L-labeling of EP_n is known for $n \geq 4$.

4. Enumerative properties

We now investigate the enumerative properties of EP_n, defined in Section 3. In the sections that follow, all wiring diagrams are assumed to be lensless, and are considered up to motion-equivalence.

4.1. Total size $X_n = |EP_n|$

In this section, we adapt methods of [3] to prove the first two enumerative results concerning $|EP_n|$, the number of equivalence classes of critical graphs (equivalently, full wiring diagrams) of order n. There is an analogy between the stabilized-interval free (SIF) permutations of [3] and medial graphs, as follows. A permutation σ may be represented as a 2-regular graph Σ embedded in a disk with n boundary vertices. Then, σ is SIF if and only if there are no dividing lines, where here a dividing line is a line ℓ between two boundary vertices such that no edge of Σ connects vertices on opposite sides of ℓ.

To begin, we define two operations on wiring diagrams in order to build large wiring diagrams out of small, and vice versa. In both definitions, fix a lensless (but not necessarily full) wiring diagram M of order n, with boundary vertices labeled $V_1, V_2, \dots, V_n$.

Definition 4.1.1. Let $w = XY$ be a wire of M. Construct the **crossed expansion of** M **at** w, denoted $M^w_{+,c}$, as follows: add a boundary vertex V_{n+1} to M between V_1 and V_n. Let A, B denote the medial boundary vertices associated to V_{n+1}, so that the medial boundary vertices A, B, X, Y appear in that order around the circle. Then, delete w from M and replace it with the crossing wires AX, BY to form $M^w_{+,c}$. Similarly, define the **uncrossed expansion of** M **at** w, denoted $M^w_{+,u}$, to be the lensless wiring digram obtained by deleting w and replacing it with the non-crossing wires AY, BX.

Definition 4.1.2. Let V_i be a boundary vertex with associated medial boundary vertices A, B, such that we have the wires $AX, BY \in M$, and $X \neq B, Y \neq A$. Define the **refinement of** M **at** V_i, denoted M^i_-, to be the lensless wiring diagram of order $n - 1$ obtained by deleting the wires AX, BY as well as the vertices A, B, V_i, and adding the wire XY.

Each construction is well-defined up to equivalence under motions by Theorem 2.4.1. It is clear that expanding M, then refining the result at the appropriate vertex, recovers M. Similarly, refining M, then expanding the result after appropriately relabeling the vertices, recovers M if the correct choice of crossed or uncrossed is made.

Lemma 4.1.3. *Let M be a full wiring diagram, with boundary vertices $V_1, V_2, \dots, V_n$. Then:*

(a) *$M^w_{+,c}$ is full for all wires $w \in M$.*
(b) *Either $M^w_{+,u}$ is full, or otherwise $M^w_{+,u}$ has exactly one dividing line, which must have V_{n+1} as one of its endpoints.*

Proof. First, suppose for sake of contradiction that $M^w_{+,c}$ has a dividing line ℓ. If ℓ is of the form $V_i V_{n+1}$, then ℓ must exit the sector formed by the two crossed wires coming out of the medial boundary vertices associated to V_{n+1}. If this is the case, we get an intersection

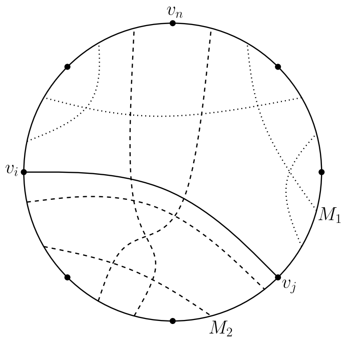


Fig. 4. $M1$ (dotted) and $M2$ (dashed), from M.

between $M_{+,c}$ and a wire, a contradiction. If instead, $\ell = V_iV_j$ with $i, j \neq n+1$, then ℓ is a dividing line in M, also a contradiction. We thus have (a). Similarly, we find that any dividing line of $M^w_{+,u}$ must have V_{n+1} as an endpoint. However, if $V_iV_{n+1}, V_{i'}V_{n+1}$ are dividing lines, then $V_iV_{i'}$ is as well, a contradiction, so we have (b). $\square$

Lemma 4.1.4. *Let M be a full wiring diagram, with boundary vertices $V_1, V_2, \dots, V_n$. Furthermore, suppose M^n_- exists and is not full. Then, M^n_- has a unique dividing line V_iV_j with $1 \leq i < j \leq n-1$ and $j-i$ maximal.*

Proof. By assumption, M^n_- has a dividing line, so suppose for sake of contradiction that $\ell_1 = V_{i_1}V_{j_1}, \ell_2 = V_{i_2}V_{j_2}$ are both dividing lines of M' with $d = j_1 - i_1 = j_2 - i_2$ maximal. Without loss of generality, assume $i_1 < i_2$ (and $i_1 < j_1, i_2 < j_2$). If $j_1 \geq i_2$, then $V_{i_1}V_{j_2}$ is also a dividing line with $j_2 - i_1 > d$, a contradiction. On the other hand, if $j_1 < i_2$, at least one of ℓ_1, ℓ_2 is a dividing line for M, again a contradiction. $\square$

If M, i, j are as above, we now define two wiring diagrams M_1 and M_2; see Fig. 4 for an example. First, let M_1 be the result of restricting M to the wires associated to the vertices V_k, for $k \in [i, j] \cup \{n\}$. Note that M_1 is a wiring diagram of order $j - i + 1$ with boundary vertices $V_i, V_{i+1}, \dots, V_j$ (and not V_n). Then, let M_2 be the wiring diagram of order $n - (j - i + 1)$ obtained by restricting M to the wires associated to the vertices V_k, for $k \notin [i, j] \cup \{n\}$.

Lemma 4.1.5. *M_1 and M_2, as above, are full.*

Proof. It is not difficult to check that any dividing line of M_1 must also be a dividing line of M, a contradiction. A dividing line $V_{i'}V_{j'}$ of M_2 must also be a dividing line of M^n_-, but then $j' - i' > j - i$, contradicting the maximality from Lemma 4.1.4. $\square$

We are now ready to prove the main theorem of this section.

Theorem 4.1.6. *Put $X_n = |EP_n|$, which here we take to be the number of full wiring diagrams of order n. Then, $X_1 = 1$, and for $n \geq 2$,*

$$X_n = 2(n-1)X_{n-1} + \sum_{k=2}^{n-2}(k-1)X_k X_{n-k}.$$

Proof. $X_1 = 1$ is obvious. For $n > 1$, we would like to count the number of full wiring diagrams M of order n, whose boundary vertices are labeled $V_1, V_2, \ldots, V_n$, in clockwise order, with medial boundary vertices A_i and B_i at each vertex, so that the order of points on the circle is A_i, V_i, B_i in clockwise order. If $A_n B_n$ is a wire, constructing the rest of M amounts to constructing a full wiring diagram of order $n-1$, so there are X_{n-1} such full wiring diagrams in this case.

Otherwise, consider the refinement M^{n}_{-}. All M for which M^{n}_{-} is full can be obtained by expanding at one of the $n-1$ wires of a full wiring diagram M' of order $n-1$. By Lemma 4.1.3, the expanded wiring diagram is full unless it has exactly one dividing line $V_k V_n$, and furthermore it is easy to see that any such graphs is an expansion of a full wiring diagram of order $n-1$.

There are $2(n-1)$ ways to expand M', and each expansion gives a different wiring diagram of order n, for $2(n-1)X_{n-1}$ total expanded wiring diagrams. However, by the previous paragraph, the number of these which are not full is $\sum_{k=1}^{n-1} X_k X_{n-k}$, as imposing a unique dividing line $V_k V_n$ forces us to construct two full wiring diagrams on either side, of orders $k, n-k$ respectively. Thus, we have $2(n-1)X_{n-1} - \sum_{k=1}^{n-1} X_k X_{n-k}$ full wiring diagrams of order n such that refining at V_n gives another full wiring diagram.

It is left to count those M such that contracting at V_n leaves a non-full wiring diagram M'. By Lemma 4.1.5, such an M gives us a pair of full wiring diagrams of orders $i+j+1, n-(i+j+1)$, where $V_i V_j$ is as in Lemma 4.1.4. Conversely, given a pair of boundary vertices $V_i, V_j \neq V_n$ of M and full wiring diagrams of orders $j-i+1, n-(j-i+1)$, we may reverse the construction $M \mapsto (M_1, M_2)$ to get a wiring diagram of order n: furthermore, it is not difficult to check that this wiring diagram is full.

It follows that the number of such M is

$$\sum_{1 \leq i < j \leq n-1} X_{j-i+1} X_{n-(j-i+1)} = \sum_{k=1}^{n-2} k X_k X_{n-k}.$$

Summing our three cases together, we find

$$\begin{aligned} X_n &= X_{n-1} + 2(n-1)X_{n-1} - \sum_{k=1}^{n-1} X_k X_{n-k} + \sum_{k=1}^{n-2} k X_k X_{n-k} \\ &= 2(n-1)X_{n-1} + \sum_{k=2}^{n-2}(k-1)X_k X_{n-k}, \end{aligned}$$

using the fact that $X_1 = 1$. The theorem is proven. $\square$

Remark 4.1.7. The sequence $\{X_n\}$ is A111088 in the OEIS [14].

We also have an analogue of the other main result of [3]; the method of proof can be readily mimicked.

Theorem 4.1.8. *Let $X(t) = \sum_{n=0}^{\infty} X_n t^n$ be the generating function for the sequence $\{X_n\}$, where we take $X(0) = 1$. Then, we have $[t^{n-1}]X(t)^n = n \cdot (2n-3)!!$.*

4.2. Asymptotic behavior of $X_n = |EP_n|$

In this section, we adapt methods from [21] to prove:

Theorem 4.2.1. *We have*

$$\lim_{n\to\infty} \frac{X_n}{(2n-1)!!} = \frac{1}{\sqrt{e}}.$$

In other words, the density of full wiring diagrams in the set of all wiring diagrams is $e^{-1/2}$. A key ingredient in the proof is the following:

Lemma 4.2.2. *For $n \geq 6$, $(2n-1)X_{n-1} < X_n < 2nX_{n-1}$.*

Proof. See Appendix A. □

To prove Theorem 4.2.1, we will estimate the number of non-full wiring graphs of order n. Let D_n denote the number of wiring diagrams formed in the following way: for $1 \leq j \leq n-2$, choose j pairs of adjacent boundary vertices, and for each pair, connect the two medial boundary vertices between them. Then, with the remaining $2n-2j$ vertices, form a full wiring diagram of order $n-j$, which in particular has no dividing lines whose endpoints are adjacent boundary vertices. It is clear that all such diagrams are non-full.

For completeness, we will also include in our count the wiring diagram where all pairs of adjacent boundary vertices give dividing lines, but because we are interested in the asymptotic behavior of D_n, this addition will be of no consequence. It is easily seen that

$$D_n = 1 + \sum_{j=1}^{n-2} \binom{n}{j} X_{n-j}.$$

Now, let E_n be the number of non-full wiring diagrams not constructed above. Consider the following construction: choose an ordered pair of distinct, non-adjacent boundary vertices on our boundary circle. Then, on each side of the directed segment, construct any wiring diagram. This construction yields

$$Y_n = n \sum_{j=2}^{n-2} (2n-2j-1)!!(2j-1)!!$$

total (not necessarily distinct) wiring diagrams, which clearly overcounts E_n.

We now state two additional lemmas, whose proofs are purely analytic and may be found in Appendix A.

Lemma 4.2.3. $D_n/X_n \to \sqrt{e} - 1$ *as* $n \to \infty$.

Lemma 4.2.4. $Y_n/X_n \to 0$ *as* $n \to \infty$.

From here, we will be able to establish the desired asymptotic.

Proof of Theorem 4.2.1. X_n, D_n, and E_n together count the total number of wiring diagrams, which is equal to $(2n-1)!!$. Thus,

$$\frac{(2n-1)!!}{X_n} = \frac{X_n + D_n + E_n}{X_n} \to 1 + (\sqrt{e} - 1) + 0 = e^{1/2},$$

assuming Lemmas 4.2.3 and 4.2.4 (we have $Y_n/X_n \to 0$, so $E_n/X_n \to 0$ as well), so the desired conclusion is immediate from taking the reciprocal. □

Let us summarize now the results of the last two sections:

Theorem 4.2.5.

(a) $X_1 = 1$ *and*

$$X_n = 2(n-1)X_{n-1} + \sum_{k=2}^{n-2} (k-1) X_k X_{n-k}.$$

(b) $[t^{n-1}]X(t)^n = n \cdot (2n-3)!!$, *where* $X(t)$ *is the generating function for the sequence* $\{X_i\}$.

(c) $X_n/(2n-1)!! \to e^{-1/2}$ *as* $n \to \infty$.

To conclude this section, we propose the following generalization of Theorem 4.2.5:

Conjecture 4.2.6. *Let* $\lambda \neq 0$ *be a real number. Consider the sequence* $\{X_{n,\lambda}\}$ *defined by* $X_{0,\lambda} = X_{1,\lambda} = 1$, *and*

$$X_n = \lambda(n-1)X_{n-1,\lambda} + \sum_{k=2}^{n-2} (k-1) X_{k,\lambda} X_{n-k,\lambda}.$$

Let $X_\lambda(t)$ *be the generating function for the sequence* $\{X_{n,\lambda}\}$. *Then,*

$$\left[t^{n-1}\right] X_\lambda(t)^n = n \cdot (1/\lambda)_n$$

and

$$\lim_{n\to\infty} \frac{X_{\lambda,n}}{(1/\lambda)_n} = \frac{1}{\sqrt[n]{e}},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$.

For $\lambda \in \mathbb{Z}$, a proof exhibiting and exploiting a combinatorial interpretation for the sequence $\{X_{n,\lambda}\}$ would be most desirable, as we have done for $\lambda = 2$. However, no such interpretation is known for $\lambda > 2$. The case $\lambda = 1$ is handled in [3] and [18, §3], though the latter does not use the interpretation of $X_{n,1}$ as SIF permutations of $[n]$ to obtain the asymptotic.

Interestingly, if we define $X_{n,-1}$ analogously, we get $X_{n,-1} = (-1)^{n+1}C_n$, where C_n denotes the n-th Catalan number, see [14].

4.3. Rank sizes $|EP_{n,r}|$

Proposition 4.3.1. *For non-negative* $c \le n-2$*, we have* $|EP_{n,\binom{n}{2}-c}| = \binom{n-1+c}{c}$*. Furthermore,* $|EP_{n,\binom{n}{2}-(n-1)}| = \binom{2n-2}{n-1} - n$*.*

Proof. For convenience, put $N = \binom{n}{2}$. We claim that for $c \le n-2$, any wiring diagram of order n with $N - c$ crossings is necessarily full. Thus, for $c \le n-2$, it suffices to compute the number of circular wiring diagrams with $N - c$ crossings. By [17, (1)] and [17, p. 218], this number is $\binom{n+c-1}{n-1}$ for $c \le n-1$. This immediately gives the desired result for $c \le n-2$, and from here the case $c = n-1$ may be handled easily. □

Proposition 4.3.1 gives an exact formula for $|EP_{n,r}|$ for r large, but no general formula is known for general r. To conclude this section, we cannot resist making the following conjecture, which has been verified up to $n = 8$.

Conjecture 4.3.2. EP_n *is rank-unimodal.*

5. Electrical positroids

By Theorem 2.2.6, $n \times n$ response matrices are characterized in the following way: a square matrix M is the response matrix for an electrical network (Γ, γ) if and only if M is symmetric, its row and column sums are zero, and its circular minors $M(P;Q)$ are non-negative. Furthermore, $M(P;Q)$ is positive if and only if there is a connection from P to Q in Γ. The sets S of circular pairs for which there exists a response matrix M with $M(P;Q)$ is positive if and only if $(P;Q) \in S$, then, are thus our next objects of study.

The case of the totally nonnegative Grassmannian was studied in [16]: for $k \times n$ (with $k < n$) matrices with non-negative maximal minors, the possible sets of positive maximal minors are called *positroids*, and are a special class of matroids. Our objects will be called *electrical positroids*, which we first construct axiomatically, then prove are exactly those sets S of positive circular minors in response matrices.

5.1. Grassmann–Plücker relations and electrical positroid axioms

Here, we present the axioms for electrical positroids, which arise naturally from the Grassmann–Plücker relations.

Definition 5.1.1. Let M be a fixed matrix, whose rows and columns are indexed by some sets I, J. We write $\Delta^{i_1 i_2 \cdots i_m, j_1 j_2 \cdots j_n}$ for the determinant of the matrix M' formed by deleting the rows corresponding to $i_1, i_2, \dots, i_m \in I$ and $j_1, j_2, \dots, j_n \in J$, provided M' is square.

While the meaning $\Delta^{i_1 i_2 \cdots i_m, j_1 j_2 \cdots j_n}$ depends on the underlying sets I, J, these sets will always be implicit.

Proposition 5.1.2. *We have the following two* ***Grassmann–Plücker*** *relations.*

(a) *Let M be an $n \times n$ matrix, with a, b elements of its row set and c, d elements of its column set. Furthermore, suppose that the row a appears above row b and column c appears to the left of column d. Then,*

$$\Delta^{a,c}\Delta^{b,d} = \Delta^{a,d}\Delta^{b,c} + \Delta^{ab,cd}\Delta^{\emptyset,\emptyset}. \tag{5.1.1}$$

(b) *Let M be an $(n+1) \times n$ matrix, with a, b, c elements of its row set (appearing in this order, from top to bottom), and let d an element of its column set. Then,*

$$\Delta^{b,\emptyset}\Delta^{ac,d} = \Delta^{a,\emptyset}\Delta^{bc,d} + \Delta^{c,\emptyset}\Delta^{ab,d}. \tag{5.1.2}$$

While the Grassmann–Plücker relations are purely algebraic in formulation, they encode combinatorial information concerning the connections of circular pairs in a circular planar graph Γ. As a simple example, consider four boundary vertices a, b, d, c in clockwise order of an electrical network (Γ, γ), and let $\pi = \pi(\Gamma)$. If M is the response matrix of (Γ, γ), then $M' = M(\{a, b\}, \{c, d\})$ is the circular minor associated to the circular pair $(a, b; c, d)$; thus, M' has non-negative determinant. Furthermore, the entries of M' are 1×1 circular minors of M, so they, too, must be non-negative.

Now, suppose that the left hand side of (5.1.1) is positive, that is, $\Delta^{a,c}\Delta^{b,d} > 0$. Equivalently, there are connections between b and d and between a and c in Γ. Then, at least one of the two terms on the right hand side must be strictly positive; combinatorially, this means that either there are connections between b and c and between a and d, or there is a connection between $\{a, b\}$ and $\{c, d\}$. One can derive similar combinatorial rules by assuming one of the terms on the right hand side is positive, and deducing that the left hand side must be positive as well.

The first six of the electrical positroid axioms given in Definition 5.1.3 summarize all of the information that can be extracted in this way from the Grassmann–Plücker relations.

We will require a few pieces of notation. If $a \in P$, write $P - a$ for the ordered set formed by removing a from P. If $a \notin P = \{a_1, a_2, \ldots, a_N\}$, write $P + a$ for the ordered set formed by adding a to the end of P. Note that this implies we can only add a to P if a_N, a, a_1 lie in clockwise order on the circle.

Definition 5.1.3. A set S of circular pairs is an **electrical positroid** if it satisfies the following eight axioms:

1. For ordered sets $P = \{a_1, a_2, \ldots, a_N\}$ and $Q = \{b_1, b_2, \ldots, b_N\}$, with $a_1, \ldots, a_N$, $b_N, \ldots, b_1$ in clockwise order (that is, $(P; Q)$ is a circular pair), consider any $a = a_i$, $b = a_j$, $c = b_k$, $d = b_\ell$ with $i < j$ and $k < \ell$. Then:
   (a) If $(P - a; Q - c), (P - b; Q - d) \in S$, then either $(P - a; Q - d), (P - b; Q - c) \in S$ or $(P - a - b; Q - c - d), (P; Q) \in S$.
   (b) If $(P - a; Q - d), (P - b; Q - c) \in S$, then $(P - a; Q - c), (P - b; Q - d) \in S$.
   (c) If $(P - a - b; Q - c - d), (P; Q) \in S$, then $(P - a; Q - c), (P - b; Q - d) \in S$.
2. For $P = \{a_1, a_2, \ldots, a_{N+1}\}$ and $Q = \{b_1, b_2, \ldots, b_N\}$, with $a_1, a_2, \ldots, a_{N+1}, b_N, \ldots, b_1$ in clockwise order, consider any $a = a_i$, $b = a_j$, $c = a_k$, $d = b_\ell$ with $i < j < k$. Then:
   (a) If $(P - b; Q), (P - a - c; Q - d) \in S$, then either $(P - a; Q), (P - b - c; Q - d) \in S$ or $(P - c; Q), (P - a - b; Q - d) \in S$.
   (b) If $(P - a; Q), (P - b - c; Q - d) \in S$, then $(P - b; Q), (P - a - c; Q - d) \in S$.
   (c) If $(P - c; Q), (P - a - c; Q - d) \in S$, then $(P - b; Q), (P - a - c; Q - d) \in S$.

Finally:

3. (**Subset axiom**) For $P = \{a_1, a_2, \ldots, a_n\}$ and $Q = \{b_1, b_2, \ldots, b_n\}$ with $(P; Q)$ a circular pair, if $(P; Q) \in S$, then $(P - a_i; Q - b_i) \in S$.
4. $(\emptyset; \emptyset) \in S$.

5.2. The main theorem

We now state the main theorem relating electrical positroids and electrical networks, and sketch the proof. Full details are given in Appendix B.

Theorem 5.2.1. *A set S of circular pairs is an electrical positroid if and only if there exists a response matrix whose positive circular minors are exactly those corresponding to S.*

Proof. Given a response matrix M, it is straightforward to check that the set S of circular pairs corresponding to the positive circular minors of M satisfies the first six axioms, by

Proposition 5.1.2. S also satisfies the Subset axiom, by Theorem 2.2.6. Finally, adopting the convention that the empty determinant is equal to 1, we have the last axiom. To prove Theorem 5.2.1, we thus need to show that any electrical positroid S may be realized as the set of positive circular minors of a response matrix, or equivalently the set of connections in a circular planar graph.

Fix a boundary circle with n boundary vertices, which we label $1, 2, \ldots, n$ in clockwise order. We have shown, via the Grassmann–Plücker relations, that the set of circular pairs corresponding to the positive circular minors of a response matrix is an electrical positroid. We now prove that, for all electrical positroids S, there exists a critical graph G for which $\pi(G) = S$, which will establish Theorem 5.2.1. The idea of the argument is as follows.

Assume, for sake of contradiction, that there exists some electrical positroid S for which there does not exist such a critical graph G with $\pi(G) = S$. Then, let S_0 have maximal size among all such electrical positroids. Note that S_0 does not contain all circular pairs $(P; Q)$, because otherwise $S_0 = \pi(G_{\max})$, where $G_{\max}$ denotes a critical representative of the top-rank element of EP_n.

We will then add circular pairs to S_0 according to the boundary edge and boundary spike properties (cf. [4, §4]), as defined below, to form an electrical positroid S_1. By the maximality of S_0, $S_1 = \pi(G_1)$ for some critical graph G_1. We will then delete a boundary edge or contract a boundary spike in G_1 to obtain a graph G_0, with $S_0' = \pi(G_0)$, and show that $\pi(G_0) = S_0$.

Definition 5.2.2. A set S of circular pairs is said to have the $(i, i+1)$**-BEP (boundary edge property)** if, for all circular pairs $(P; Q) \in S$, if $(P + i; Q + (i + 1))$ is a circular pair, then $(P + i; Q + (i + 1)) \in S$. A set S of circular pairs is said to have the i**-BSP (boundary spike property)** if, for any circular pairs $(P; Q) \in S$ and x, y such that $(P + x; Q + i), (P + i; Q + y) \in S$, we have $(P + x; Q + y) \in S$.

It is easy to see that the properties we defined correspond to graphs having a boundary edge or boundary spike. We next have the following lemma:

Lemma 5.2.3. *If S has all n BEPs and all n BSPs, then S contains all circular pairs.*

Proof. We proceed by induction on the size of $(P; Q)$ that $(P; Q) \in S$ for all circular pairs $(P; Q)$. First, suppose that $|P| = 1$. First, $(i; i + 1) \in S$ for all i, because it has all BEPs and $(\emptyset; \emptyset) \in S$. Then, because S has the i-BSP, and $(i - 1; i), (i; i + 1) \in S$ we obtain $(i - 1; i + 1) \in S$. Continuing in this way gives that S contains all circular pairs $(P; Q)$ with $|P| = 1$.

Now, suppose that S contains all circular pairs of size $k-1$. Let $(a_1, \ldots, a_k; b_1, \ldots, b_k)$ be a circular pair of size k. By assumption, $(a_2, \ldots, a_k; b_2, \ldots, b_k) \in S$. Because S has all BEPs, $(b_1 + 1, a_2, \ldots, a_k; b_1, b_2, \ldots, b_k) \in S$ and $(b_1 + 2, a_2, \ldots, a_k; b_1 + 1, b_2, \ldots, b_k) \in S$, so by the $(b_1 + 1)$-BSP, $(b_1 + 2, a_2, \ldots, a_k; b_1, b_2, \ldots, b_k) \in S$. Continuing in this way gives $(a_1, \ldots, a_k; b_1, \ldots, b_k) \in S$, so we have the desired claim. □

Lemma 5.2.3 tells us that we may assume S_0 does not have all BEPs and BSPs. We first assume that S_0 does not have some BEP, without loss of generality the $(n, 1)$-BEP. We will now add circular pairs to S_0 to obtain an electrical positroid S_1 that does have the $(n, 1)$-BEP. Specifically, we add to S_0 every circular pair $(P + 1; Q + n)$, where $(P; Q) \in S_0$ has $1 < a_1 < b_1 < n$ (here $P = \{a_1, \ldots, a_k\}, Q = \{b_1, \ldots, b_k\}$), to obtain S_1. It is easy to check that S_1 is an electrical positroid.

By assumption, S_0 is the maximal electrical positroid for which any circular planar graph G has $\pi(G) \neq S_0$. Thus, there exists a graph G_1 be a graph such that $\pi(G_1) = S_1$, and G_1 may be taken to have a boundary edge $(n, 1)$. Then, let G_0 be the result of deleting the boundary edge $(n, 1)$, and $S_0' = \pi(G_0)$. To obtain a contradiction, it is enough to prove that $S_0 = S_0'$.

We first give a definition.

Definition 5.2.4. Consider a circular pair $(P; Q) \in S_0$ for which $1, n \notin P \cup Q$. We will assume, for the rest of this section, that $(P + 1; Q + n)$ is a circular pair. $(P; Q)$ is said to be is **incomplete** if $(P + 1; Q + n) \notin S_0$, and **complete** if $(P + 1; Q + n) \in S_0$.

Then, the first step in proving $S_0 = \pi(G_0)$ lies in the following lemma:

Lemma 5.2.5. *For any two incomplete circular pairs $(P; Q)$ and $(P'; Q')$, any electrical positroid Z satisfying $S_0 \cup \{(P+1; Q+n)\} \subset Z \subset S_1$ must also contain $(P'+1; Q'+n)$.*

The idea of the proof is to apply the axioms repeatedly, but the details are cumbersome, so we defer this pursuit to Appendix B. Then, by the above result, if we start with our set S_0 and some incomplete circular pair $(P; Q) \in S_0$, "completing" $(P; Q)$ by adding $(P + 1; Q + n)$ to S_0 will require that we have completed every incomplete pair. We now finish the proof of Theorem 5.2.1, in the boundary edge case.

Let $T_0 \subset S_0$ denote the subset of circular pairs in S_0 without the connection $(1, n)$, and define T_1, T_0' similarly for S_1, S_0', respectively. By construction, it is easily seen that $T_0 = T_1 = T_0'$. While T_0 may not necessarily be an electrical positroid, we have:

Lemma 5.2.6. *There exists an electrical positroid T with $T_0 \subset T \subset S_0 \cap S_0'$.*

Proof. Deferred to Appendix B. $\square$

We now complete the proof of the main theorem.

By Lemma 5.2.5, we must in fact have $S_0 = T = S_0'$, provided that neither S_0 nor S_0' is equal to S_1, which is true by construction (recall that G_0' is critical). The proof is complete, in the boundary edge case.

It is left to consider the case in which S_0 has the $(i, i + 1)$-BEP, for each i, but fails to have the i-BSP, for some i. Without loss of generality, suppose that S_0 does not have the 1-BSP. We now form S_1 as the union of S_0 and the set of all circular pairs

$(P+x;Q+y)$ such that $(P+x;Q+1), (P+1,Q+y) \in S_0$, where $(P;Q)$ is a circular pair with $1, x \notin P, 1, y \notin Q$.

As in the boundary edge case, we form a circular planar graph G_1 such that $\pi(G_1) = S_1$ and G_1 has a boundary spike at 1. Then, contracting this boundary spike to obtain the graph G_0, we find that $\pi(G_0) = S_0$. The details are the same as in the boundary edge case, so they are omitted. $\square$

6. The LP algebra $\mathcal{LM}_n$

We now study the LP Algebra $\mathcal{LM}_n$. Our starting point will be *positivity tests*; a particular positivity test will form the initial seed in $\mathcal{LM}_n$. We then proceed to investigate the algebraic and combinatorial properties of clusters in $\mathcal{LM}_n$.

6.1. Positivity tests

Let M be a symmetric $n \times n$ matrix with row and column sums equal to zero. In this section, we describe tests for deciding if M is the response matrix for an electrical network Γ in the top cell of EP_n. That is, we describe tests for deciding if all of the circular minors of M are positive. These tests are similar to certain tests for total positivity described in [8]. Throughout the remainder of this section, all indices around the circle are considered modulo n, and we will refer to circular pairs and their corresponding minors interchangeably.

Definition 6.1.1. A set S of circular pairs is a **positivity test** if, for all matrices M whose minors corresponding to S are positive, every circular minor of M is positive (equivalently, M is the response matrix for a top-rank electrical network).

We begin by describing a positivity test of size $\binom{n}{2}$. Fix n vertices on a boundary circle, labeled $1, 2, \ldots, n$ in clockwise order.

Definition 6.1.2. For two points $a, b \in [n]$, let $d(a, b)$ denote the number of boundary vertices on the arc formed by starting at a and moving clockwise to b, inclusive.

Definition 6.1.3. A circular pair $(P;Q) = (p_1, \cdots, p_k; q_1, \cdots, q_k)$ is called **solid** if both sequences $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$ appear consecutively in clockwise order around the circle. Write $d_1 = d_1(P;Q) = d(p_k, q_k)$, and $d_2 = d_2(P;Q) = d(q_1, p_1)$. We will call a solid circular pair $(P;Q)$ **picked** if one of the following conditions holds:

- $d_1 \leq d_2$ and $1 \leq p_1 \leq \frac{n}{2}$, or
- $d_1 \geq d_2$ and $1 \leq q_k \leq \frac{n}{2}$

Definition 6.1.4. Let M be a fixed symmetric $n \times n$ matrix. Define the set of **diametric pairs** $\mathcal{D}_n$ to be the set of solid circular pairs $(P;Q)$ such that either $|d_1 - d_2| \leq 1$ or

$|d_1 - d_2| = 2$ and $(P; Q)$ is picked. We will refer to the elements of $\mathcal{D}_n$ as circular pairs and minors interchangeably.

It is easily checked that $|\mathcal{D}_n| = \binom{n}{2}$.

Remark 6.1.5. For a solid circular pair $(P; Q)$, we have that $|d_1 - d_2| \equiv n \pmod 2$, so $\mathcal{D}_n$ consists of the solid circular pairs with $|d_1 - d_2| = 1$ when n is odd, and the solid circular pairs with either $|d_1 - d_2| = 0$, or $|d_1 - d_2| = 2$ and $(P; Q)$ is picked when n is even.

Recall (see Remark 2.2.2) that the circular pairs $(P; Q)$ and $(\widetilde{Q}; \widetilde{P})$ will be regarded as the same. Note, for example, that $(P; Q) \in \mathcal{D}_n$ if and only if $(\widetilde{Q}; \widetilde{P}) \in \mathcal{D}_n$, so the definition of $\mathcal{D}_n$ is compatible with this convention.

Proposition 6.1.6. *If M is taken to be an $n \times n$ symmetric matrix of indeterminates, any circular minor is a positive rational expression in the determinants of the elements of $\mathcal{D}_n$.*

Proof. See [10, Theorem 4.12]. $\square$

Corollary 6.1.7. *$\mathcal{D}_n$ is a positivity test.*

6.2. $\mathcal{CM}_n$ and $\mathcal{LM}_n$

The positive rational expressions from the previous section are reminiscent of a cluster algebra structure (see [7, §3] for definitions). In fact, (5.1.1) and (5.1.2) are exactly the exchange relations for the local moves in double wiring diagrams [8, Fig. 9]. Due to parity issues similar to those encountered when attempting to associate a cluster algebra to a non-orientable surface in [6], the structure of positivity tests is slightly different from a cluster algebra. We present the structure in two different ways: first, as a Laurent phenomenon (LP) algebra $\mathcal{LM}_n$ (see [12, §2,3] for definitions), and secondly as a cluster algebra $\mathcal{CM}_n$ similar to the double cover cluster algebra in [6]. $\mathcal{LM}_n$, we will find, is isomorphic to the polynomial ring on $\binom{n}{2}$ variables, but more importantly encodes the information of the positivity of the circular minors of a fixed $n \times n$ matrix.

We begin by describing an undirected graph U_n that encodes the desired mutation relations among our initial seed. The vertex set of U_n will be $V_n = \mathcal{D}_n \cup \{(\emptyset; \emptyset)\}$.

Definition 6.2.1. A solid circular pair $(p_1, \ldots, p_k; q_1, \ldots, q_k)$ is called **maximal** if $2k+2 > n$ or $2k + 2 = n$ and $d_1 = d_2$. A solid circular pair $(P; Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ is called **limiting** if $|d_1 - d_2| = 2$, $(P; Q)$ is picked, and $1 = p_1$ or $1 = q_k$.

Let us now describe the edges of U_n: see Fig. 5 for an example. For each $(P; Q) \in V_n$ that is not maximal, limiting, or empty (that is, equal to $(\emptyset; \emptyset)$), there is a unique way to

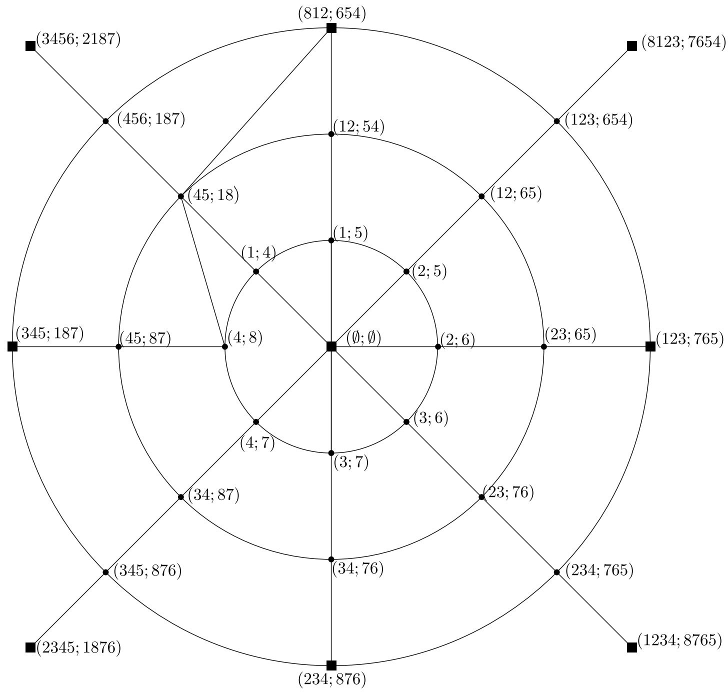


Fig. 5. The graph U_8 depicting the desired exchange relations among $\mathcal{D}_8$. Vertices marked as squares correspond to frozen variables. $(4;8)$, $(45;18)$ and $(812;654)$ are the limiting circular pairs.

substitute values in (5.1.1) such that $(P;Q)$ appears on the left hand side, and all four terms on the right hand side are in V_n. We draw edges from $(P;Q)$ to these four vertices in U_n. Finally, if $(P;Q), (R;S) \in V_n$ are limiting, we draw an edge between them if their sizes differ by 1. The edges drawn in these two cases constitute all edges of V_n.

For any maximal circular pair $(P;Q)$, it can be proven that there exists a symmetric matrix A such that A is positive on any circular pair except $A(P;Q) \le 0$. In fact, if $(P;Q)$ is maximal and has $|d_1 - d_2| \le 1$, then the set of all circular pairs other than $(P;Q)$ is an electrical positroid. Hence, in our quivers, we will take the vertices corresponding to the maximal circular pairs and $(\emptyset;\emptyset)$ to be frozen.

U_n can then be embedded in the plane in a natural way with the circular pairs of size k lying on the circle of radius k centered at $(\emptyset;\emptyset)$, and all edges either along those circles or radially outward from $(\emptyset;\emptyset)$, except for the edges between vertices corresponding to limiting circular pairs. Fig. 5 demonstrates this embedding for $n = 8$.

If we could orient the edges of U_n such that they alternate between in- and out-edges at each non-frozen vertex, then the resulting quiver would give a cluster algebra such that mutations at vertices whose associated cluster variables are neither frozen not limiting correspond to the relation Grassmann–Plücker relation (5.1.1). Furthermore, these mutation relations among the vertices of V_n constitute all of the Grassmann–Plücker relations for which five of the six terms on the right hand side are elements in V_n, and the term which is not in V_n is on the left hand side of the relation (5.1.1) or (5.1.2).

However, for $n \geq 5$, such an orientation of the edges of U_n is impossible, because the dual graph of U_n contains odd cycles. We thus define:

Definition 6.2.2. Let $\mathcal{LM}_n$ be the LP algebra constructed as follows: the initial seed $\mathcal{S}_n$ has cluster variables equal to the minors in V_n, with the maximal pairs and $(\emptyset;\emptyset)$ frozen, and, for any other $(P;Q) \in V_n$, the exchange polynomial $F_{(P;Q)}$ is the same as what is obtained from a quiver with underlying graph U_n, such that the edges around the vertex associated to $(P;Q)$ in U_n alternate between in- and out-edges.

For example, in $\mathcal{LM}_8$, the exchange polynomial associated to the cluster variable $x_{(12;54)}$ is $x_{(45;18)}x_{(12;65)}+x_{(1;5)}x_{(812;654)}$. We need the additional technical condition that $F_{(P;Q)}$ is irreducible as a polynomial in the cluster variables V_n, but the irreducibility is clear.

We next define a cluster algebra $\mathcal{CM}_n$ which is a double cover of positivity tests, in the following sense: we begin by considering an $n \times n$ matrix M', which we no longer assume to be symmetric. We write **non-symmetric circular pairs** in the row and column sets of M' as $(P;Q)'$, so that $(P;Q)'$ and $(\widetilde{Q};\widetilde{P})'$ now represent different circular pairs. We will say that two expressions A, B in the entries of M' **correspond** if swapping the rows and columns for each entry in A gives B, and we will write $B = c(A)$. For instance, $(P;Q)' = c((Q;P)')$. c is be analogously defined on index sets.

The set of cluster variables V'_n in our initial seed will consist of pairs $(P;Q)'$ such that $(P;Q) \in V_n$. Note that $|V'_n| = 2\binom{n}{2}+1$, as V'_n contains $(P;Q)'$ and $(\widetilde{Q};\widetilde{P})'$ for each $(P;Q) \in \mathcal{D}_n$, and finally $(\emptyset;\emptyset)$. $(P;Q)'$ will be frozen in V'_n if $(P;Q)$ was frozen in V_n.

We construct the undirected graph U'_n with vertex set V'_n by adding edges in the same way that U_n was constructed. The only difference in our description is that if $(P;Q),(R;S) \in V_n$ are limiting, then they will be adjacent only if their sizes differ by 1 and $P \cap R \neq \emptyset$. See Fig. 6 for an example.

Unlike in U_n, the edges of U'_n can be oriented such that they are alternating around each non-frozen vertex. Let $\mathcal{Q}_n$ be the quiver from either orientation. Then, let $\mathcal{CM}_n$ be the cluster algebra with initial quiver $\mathcal{Q}_n$.

Breaking the symmetry of M' removed the parity problems from U_n, so that we could define a cluster algebra, but we are still interested in using U'_n to study M when M is symmetric. Toward this goal, we can restrict ourselves so that whenever we mutate at a cluster variable v, we then mutate at $c(v)$ immediately afterward. Call this restriction the **symmetry restriction**.

Lemma 6.2.3. *After the mutation sequence $\mu_{x_1}, \mu_{c(x_1)}, \mu_{x_2}, \mu_{c(x_2)}, \ldots, \mu_{x_r}, \mu_{c(x_r)}$ from the initial seed in $\mathcal{CM}_n$, the number of edges from x to y in the quiver is equal to the number of edges from $c(y)$ to $c(x)$ for each x, y in the final quiver.*

Proof. We proceed by induction on r; for $r = 0$, we have the claim by construction. Now, suppose that we have performed the mutations $\mu_{x_1}, \mu_{c(x_1)}, \mu_{x_2}, \mu_{c(x_2)}, \ldots, \mu_{x_{r-1}}, \mu_{c(x_{r-1})}$

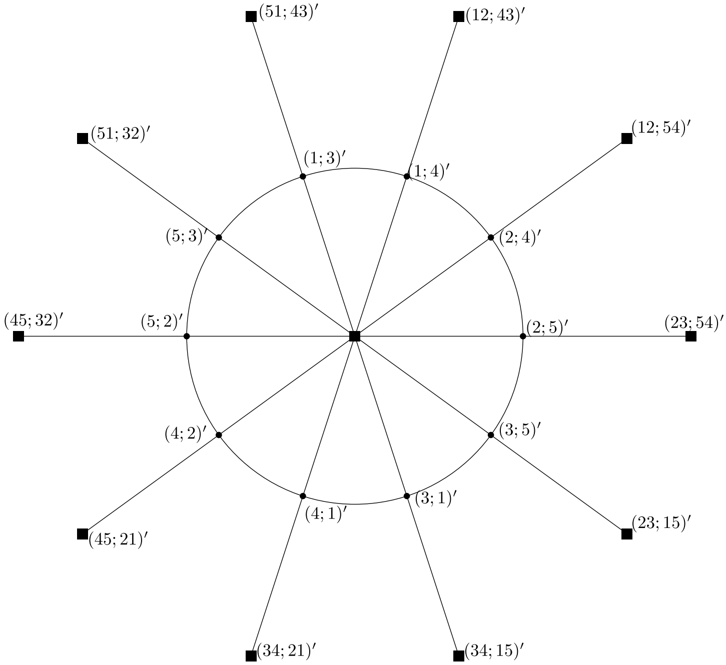


Fig. 6. The graph U_5'. In the quiver $\mathcal{Q}_n$, edges alternate directions around a non-frozen vertex.

and currently have the desired symmetry property. By the inductive hypothesis, x_r and $c(x_r)$ are not adjacent, or else we would have had edges between them in both directions, which would have been removed after mutations. Thus, no edges incident to $c(x_r)$ are created or removed upon mutating at x_r. Hence, mutating at $c(x_r)$ afterward makes the symmetric changes to the graph, as desired. $\square$

Definition 6.2.4. Let $\mathbb{C}[M]$ and $\mathbb{C}[M']$ denote the polynomial rings in the off-diagonal entries of M and M' respectively; recall that M is symmetric, so $M_{ij} = M_{ji}$. Then, we can define the **symmetrizing homomorphism** $C : \mathbb{C}[M'] \to \mathbb{C}[M]$ by its action on the off-diagonal entries of M':

$$C\left(M'_{ij}\right) = C\left(M'_{ji}\right) = M_{ij}.$$

It is the homomorphism induced by the inclusion of the polynomial ring on the entries of symmetric matrices into that of all matrices. If S is a set of polynomials in $\mathbb{C}[M']$, then write $C(S) = \{C(s) \mid s \in S\}$.

Lemma 6.2.5. *Let L_1' be the cluster of $\mathcal{CM}_n$ that results from starting at the initial cluster and performing the sequence of mutations $\mu_{x_1}, \mu_{c(x_1)}, \mu_{x_2}, \mu_{c(x_2)}, \dots, \mu_{x_r}, \mu_{c(x_r)}$. Let L_2 be the cluster of $\mathcal{LM}_n$ that results from starting at the initial cluster and performing the sequence of mutations $\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_r}$. Then, $C(L_1') = L_2$.*

Proof. Using Lemma 6.2.3, the proof is similar to [12, Proposition 4.4]. □

In light of Lemma 6.2.5, we may understand the clusters in $\mathcal{LM}_n$ by forming "double-cover" clusters in $\mathcal{CM}_n$. A sequence μ of mutations in $\mathcal{LM}_n$ corresponds to a sequence μ' of twice as many mutations in $\mathcal{CM}_n$, where we impose the symmetry restriction, and the cluster variables in $\mathcal{LM}_n$ after applying μ are the symmetrizations of those in $\mathcal{CM}_n$ after applying μ'.

Lemma 6.2.6. *Any cluster S of $\mathcal{LM}_n$ consisting entirely of circular pairs is a positivity test.*

Proof. In $\mathcal{CM}_n$, the exchange polynomial has only positive coefficients, so each variable in any cluster is a rational function with positive coefficients in the variables of any other cluster. In particular, each non-symmetric circular pair in $V_n - (\emptyset;\emptyset)'$ is a rational function with positive coefficients in the variables of any cluster reachable under the symmetry restriction. Hence, by Lemma 6.2.5, each circular pair in $\mathcal{D}_n$ can be written as a rational function with positive coefficients of the variables in S. The desired result follows easily. □

As with double wiring diagrams for totally positive matrices [8], and plabic graphs for the totally nonnegative Grassmannian [16], we now restrict ourselves to certain types of mutations in $\mathcal{LM}_n$. A natural choice is mutations with exchange relations of the form 5.1.1 or 5.1.2. These mutations keep us within clusters consisting entirely of circular minors, the "Plücker clusters."

We begin by restricting ourselves only to mutations with exchange relations of the form 5.1.1. Because the initial seed $\mathcal{S}_n$ consists only of solid circular pairs, we will only be able to mutate to other clusters consisting entirely of solid circular pairs. Our goal is to characterize these clusters. We will be able to write down such a characterization using Corollary 6.2.15 and Lemma 6.2.5, and give a more elegant description of the clusters in Proposition 6.3.6.

Definition 6.2.7. Let $(P;Q)' = (p_1,\dots,p_k;q_1,\dots,q_k)'$ be a non-symmetric, non-empty circular pair. Define the statistics $D(P;Q)'$, $T(P;Q)'$, and $k(P;Q)'$ by:

$$D(P;Q)' = d_1(P;Q)' - d_2(P;Q)' = d(p_k,q_k) - d(q_1,p_1)$$

$$T(P;Q)' = \begin{cases} \frac{p_1+q_1}{2} \pmod n & \text{if } p_1 < q_1 \\ \frac{p_1+q_1+n}{2} \pmod n & \text{if } p_1 > q_1 \end{cases}$$

$$k(P;Q)' = |P|, \quad \text{that is, the size of } (P;Q)'.$$

Remark 6.2.8. A non-symmetric solid circular pair $(P;Q)'$ is uniquely determined by the triple $(D(P;Q)',T(P;Q)',k(P;Q)')$. A necessary condition for a triple (D,T,k) to correspond to a non-symmetric solid circular pair is that $|D| + 2k \leq n$. When the terms are non-symmetric solid circular pairs, (5.1.1) can be written using these triples as:

$$\begin{aligned}(D-2,T,k)(D+2,T,k) = {} & (D,T-1/2,k)(D,T+1/2,k) \\ & + (D,T,k+1)(D,T,k-1).\end{aligned} \tag{6.2.1}$$

Definition 6.2.9. We call two non-symmetric solid circular pairs corresponding to the triples (D_1, T_1, k_1) and (D_2, T_2, k_2) **adjacent** if $T_1 = T_2$ and $|k_1 - k_2| = 1$, or $k_1 = k_2$ and $T_1 - T_2 \equiv \pm 1/2 \pmod n$. We call $(P;Q)'$ and $(R;S)'$ **diagonally adjacent** if there are two non-symmetric solid circular pairs $(A;B)', (C;D)'$ which are both adjacent to both $(P;Q)'$ and $(R;S)'$. We call $(A;B)', (C;D)'$ the **connection** of $(P;Q)', (R;S)'$.

Note that, in the initial quiver $\mathcal{Q}_n$, adjacent and diagonally adjacent circular pairs correspond to vertices which are adjacent in particular ways. Specifically, adjacent circular pairs correspond to vertices which are adjacent on the same concentric circle, or along the same radial spoke of U'_n. Diagonally adjacent circular pairs correspond to those which are adjacent via all other edges, the "diagonal" edges. We can now classify clusters of $\mathcal{CM}_n$ which can be reached only using the mutations with exchange relation (5.1.1).

Definition 6.2.10. We call a set S of $2\binom{n}{2} + 1$ non-symmetric solid circular pairs a **solid cluster** if it has the following properties:

- $(\emptyset;\emptyset)' \in S$,
- for each integer $1 \leq k \leq \frac{n}{2}$, and each $T \in \{0.5, 1, 1.5, 2, \ldots, n\}$, unless $k = \frac{n}{2}$ and T is an integer, there is a D such that the non-symmetric solid circular pair corresponding to (D,T,k) is in S, and
- if $(P;Q)', (R;S)' \in S$ and $(P;Q)'$ is adjacent to $(R;S)'$, then $|D(P;Q)' - D(R;S)'| = 2$.

Remark 6.2.11. There is a natural embedding of a solid cluster S in the plane, similar to our embedding of U'_n. We place $(\emptyset;\emptyset)'$ at any point, and then pairs of size k on the circle of radius k centered at that point. Moreover, we place adjacent pairs of the same size consecutively around each circle, and adjacent pairs of different sizes collinear with $(\emptyset;\emptyset)'$.

Definition 6.2.12. For a solid cluster S of $\mathcal{CM}_n$ and associated quiver $\mathcal{B}$, we call (S,B) a **solid seed** if it has the following properties:

- vertices corresponding to maximal non-symmetric solid circular pairs are frozen,
- there is an edge between any pair of adjacent vertices that are not both frozen,
- there is an edge between diagonally adjacent vertices $(P;Q)', (R;S)'$ if their connection $(A;B)', (F;G)'$ satisfies $|D(A;B)' - D(F;G)'| = 4$,

- there is an edge from a size 1 vertex $(P; Q)'$ to $(\emptyset; \emptyset)'$ if it would make the degree of $(P; Q)'$ even,
- all edges of $\mathcal{B}$ are in drawn in one of the four ways described above, and
- all edges are oriented so that, in the embedding described in Remark 6.2.11, edges alternate between in- and out-edges around any non-frozen vertex.

If, furthermore, $s \in S$ if and only if $C(s) \in S$, or equivalently, the non-symmetric solid circular pair corresponding to (D, T, k) is in S if and only if that corresponding to $(-D, T, k)$ is, then we call (S, b) a **symmetric solid seed**.

See, for example, Fig. 7b.

Remark 6.2.13. In a solid seed $(S, \mathcal{B})$, a variable $(P; Q)' \in S$ has an exchange polynomial of the form (5.1.1) whenever its corresponding vertex in $\mathcal{B}$, and has edges to the vertices corresponding to its four adjacent variables in $\mathcal{B}$, and no other vertices.

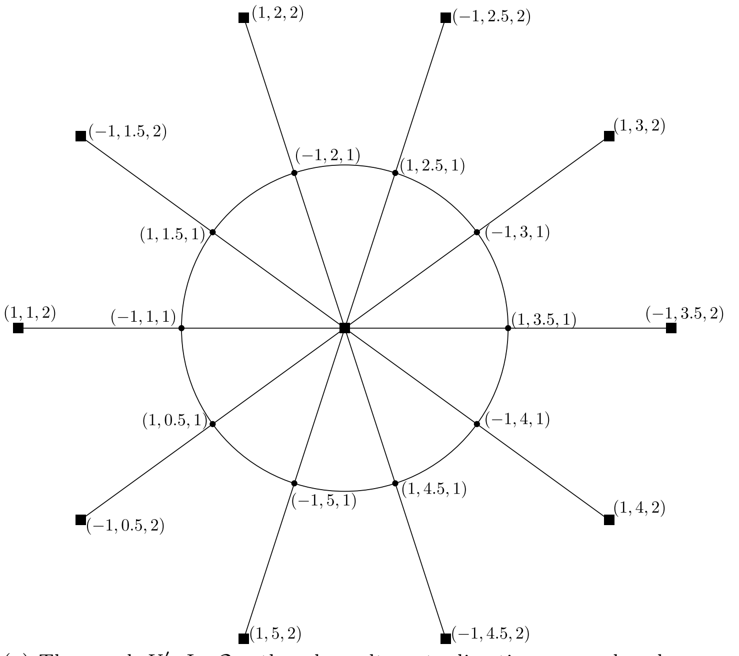
Lemma 6.2.14. *In $\mathcal{CM}_n$, from the initial seed with cluster V'_n and quiver $\mathcal{Q}'_n$, mutations of the form (5.1.1) may be applied to obtain the seed $(W'_n, \mathcal{R}'_n)$ if and only if $(W'_n, \mathcal{R}'_n)$ is a solid seed. Here, we do* not *impose the symmetry restriction.*

Proof. First, assume that $(W'_n, \mathcal{R}'_n)$ can be obtained via mutations of the form (5.1.1). First, it is easy to check that $(V'_n, \mathcal{Q}'_n)$ is a solid seed. Then, our mutations do indeed turn non-symmetric solid circular pairs into other non-symmetric solid circular pairs. Furthermore, when we perform a mutation of the form (6.2.1) at the vertex v, the values of T and k do not change, and the value of D changes from being either 2 more than the values of D at the vertices adjacent to v to being 2 less, or vice versa. Hence, the resulting seed is also solid, so, by induction, $(W'_n, \mathcal{R}'_n)$ is solid.

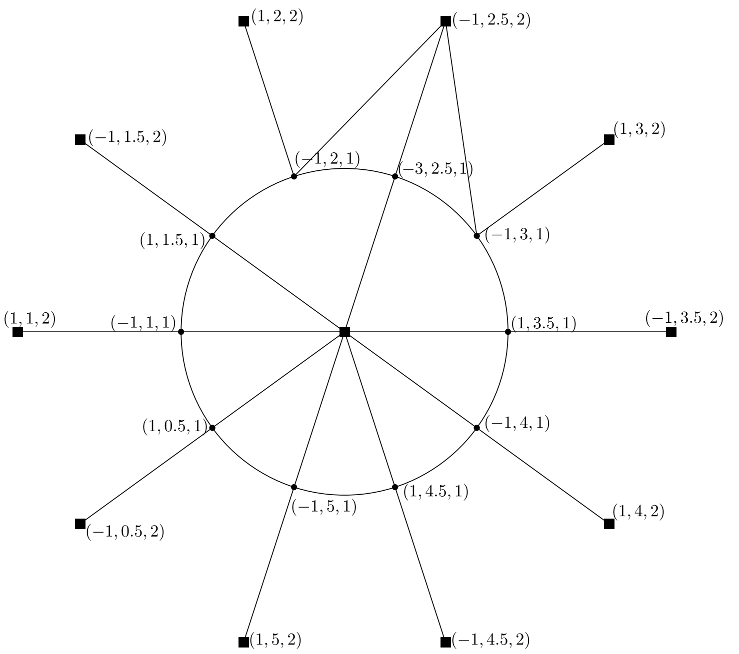
Conversely, assume $(W'_n, \mathcal{R}'_n)$ is solid. We begin by noting that, by Remark 6.2.13, whenever the four terms on the right hand side of (6.2.1) and one term on the left hand side are in our cluster, then we can perform the corresponding mutation.

Now, define $(I'_n, \mathcal{QI}'_n)$ to be the unique symmetric solid seed such that, for each $(P; Q)' \in I'_n$, $D(P; Q)' \in \{-2, -1, 0, 1, 2\}$, like the one shown in Fig. 7a. For any solid seed $(W'_n, \mathcal{R}'_n)$, we give a mutation sequence $\mu_{(W'_n, \mathcal{R}'_n)}$ using only mutations of the form (6.2.1) that transforms $(W'_n, \mathcal{R}'_n)$ into (I'_n, QI'_n). Hence, we will be able to get from the seed $(V'_n, \mathcal{Q}'_n)$ to $(W'_n, \mathcal{R}'_n)$ by performing $\mu_{(V'_n, \mathcal{Q}'_n)}$, followed by $\mu_{(W'_n, \mathcal{R}'_n)}$ in reverse order.

It is left to construct the desired mutation sequence. Again, we will be sure only to perform mutations described in Remark 6.2.13. We define $\mu_{(W'_n, \mathcal{R}'_n)}$ as follows: while the current seed is not $(I'_n, \mathcal{QI}'_n)$, choose a vertex v of the quiver, with associated cluster variable $(P; Q)'$, for which the value of $|D(P; Q)'|$ is maximized. We must have $|D(P; Q)'| > 2$, and by maximality, for each vertex $(R; S)'$ adjacent to $(P; Q)'$, we must have $|D(R; S)'| = |D(P; Q)'| - 2$. Hence, we can mutate at $(P; Q)$ to reduce $|D(P; Q)'|$ by at least 2. This process may be iterated to decrease the sum, over all cluster variables



(a) The graph U_5'. In $\mathcal{Q}_n$, the edges alternate directions around each non-frozen vertex. Compare to Fig. 6.



(b) The graph U_5' after a mutation at $(1, 2.5, 1)$. In the quiver, the edges alternate directions around each non-frozen vertex.

Fig. 7. Example of a mutation on the graph U_5' with non-symmetric solid circular pairs labeled as triples (D, T, k).

$(P;Q)'$ in our seed, of the $|D(P;Q)'|$, until we reach the seed $(I'_n, \mathcal{QI}'_n)$. The proof is complete. $\square$

Corollary 6.2.15. *In $\mathcal{CM}_n$, from our initial seed with cluster V'_n and quiver $\mathcal{Q}'_n$, we can apply symmetric pairs of mutations (that is, mutations with the symmetry restriction) of the form (5.1.1) to obtain the seed $(W'_n, \mathcal{R}'_n)$ if and only if $(W'_n, \mathcal{R}'_n)$ is a symmetric solid seed.*

Proof. The proof is almost identical to that of Lemma 6.2.14. It suffices to note that in a symmetric solid seed, $(P;Q)'$ has a maximal value of $|D|$ if and only if $c(P;Q)'$ does, so the mutation sequence $\mu_{(W'_n,\mathcal{R}'_n)}$ can be selected to obey the symmetry restriction. $\square$

Using Corollary 6.2.15 and Lemma 6.2.5, we could also prove a similar result for the LP algebra $\mathcal{LM}_n$. However, we will wait to do so until Lemma 6.3.6 where we will describe it more elegantly using the notion of weak separation.

We have now have the required machinery to prove our main theorem of this section.

Theorem 6.2.16. *Fix a symmetric $n \times n$ matrix M of distinct indeterminates. Then, the $\mathbb{C}$-Laurent phenomenon algebra $\mathcal{LM}_n$ is isomorphic to the polynomial ring (over $\mathbb{C}$) on the $\binom{n}{2}$ non-diagonal entries of M.*

Proof. We may directly apply the use of [7, Proposition 3.6] for LP algebras (for which the proof is identical) to $\mathcal{LM}_n$ as defined in Definition 6.2.2. It is well-known that minors of a matrix of indeterminates are irreducible, so we immediately have that all of our seed variables are pairwise coprime. We also need to check that each initial seed variable is coprime to the variable obtained by mutating its associated vertex in $\mathcal{Q}_n$, and that this new variable is in the polynomial ring generated by the non-diagonal entries of M. For non-limiting (and non-frozen) minors, this is clear, because each such mutation replaces a minor with another minor via (5.1.1). For limiting minors, it is not difficult to check that this is not the case.

It remains to check, then that each of the $n(n-1)$ non-diagonal entries of M' appear as cluster variables in some cluster of $\mathcal{CM}_n$. However, because 1×1 minors are solid, the result is immediate from Lemma 6.2.15 and Lemma 6.2.5. $\square$

Because all 1×1 minors are solid, mutations of the form (5.1.1) were sufficient to establish Theorem 6.2.16. Once we allow mutations of the form (5.1.2), the clusters become more difficult to describe. However, let us propose the following conjecture, which has been established computationally for $n \leq 6$.

Conjecture 6.2.17. *Every cluster of $\mathcal{LM}_n$ consisting entirely of circular pairs can be reached from the initial cluster using only mutations of the form (5.1.1) or (5.1.2).*

6.3. Weak separation

We next introduce an analogue of weakly separated sets from [13] for circular pairs. We recall the definition used in [19,15], which is more natural in this case[2]:

Definition 6.3.1. Two sets $A, B \subset [n]$ are **weakly separated** if there are no $a, a' \in A \setminus B$ and $b, b' \in B \setminus A$ such that $a < b < a' < b'$ or $b < a < b' < a'$.

Our analogue is as follows:

Definition 6.3.2. Two circular pairs $(P; Q)$ and $(R; S)$ are **weakly separated** if $P \cup R$ is weakly separated from $Q \cup S$, and $P \cup S$ is weakly separated from $Q \cup R$.

Remark 6.3.3. Note that $(P; Q)$ is weakly separated from itself and from $(\widetilde{Q}; \widetilde{P})$. Furthermore, $(P; Q)$ is weakly separated from $(R; S)$ if and only if $(\widetilde{Q}; \widetilde{P})$ is, so under the convention $(P; Q) = (\widetilde{Q}; \widetilde{P})$, weak separation is well-defined.

Conjecture 6.3.4. *Let C be a set of circular minors, for an $n \times n$ generic response matrix. Then the following are equivalent.*

- P: *C is a minimal positivity test.*
- S: *C is a maximal set of pairwise weakly separated circular pairs.*
- C: *C is a cluster of $\mathcal{LM}_n$.*

Conjecture 6.3.4 has been computationally verified for $n \leq 6$. It is not immediately clear to the authors what the correct analogues of this statement are in other total positivity settings.

We now prove various weak forms of this conjecture. First, for all clusters C of $\mathcal{LM}_n$ that are reachable from the initial seed via Grassmann–Plücker relations (cf. Conjecture 6.2.17), the elements of C are pairwise weakly separated:

Proposition 6.3.5. *If C is a set of pairwise weakly separated circular pairs such that, for some substitution of values into (5.1.1) or (5.1.2), all the terms on the right hand side, and one term $(P; Q)$ on the left hand side, are in C, then the remaining term $(R; S)$ on the left hand side is weakly separated from all of $C - (P; Q)$.*

Proof. Let a, b, c, d be as in (5.1.1) or (5.1.2). It is clear that $(R; S)$ can only be non-weakly separated from an element of $C - (P; Q)$, if a, b, c, d are boundary vertices forcing the non-weak separation.. However, this is easily seen to be impossible. $\square$

[2] Our definition varies slightly from that in the literature in the case where the two sets do not have the same size.

When restricting ourselves to clusters of solid minors, the analogue of Corollary 6.2.15 for $\mathcal{LM}_n$ matches exactly with a weak form of the equivalence $\mathsf{S} \Leftrightarrow \mathsf{C}$ in Conjecture 6.3.4.

Proposition 6.3.6. *A set C of solid circular pairs can be reached (as a cluster) from the initial cluster $\mathcal{S}_n$ in $\mathcal{LM}_n$ using only mutations of the form (5.1.1) if and only if C is a set of $\binom{n}{2}$ pairwise weakly separated solid circular pairs.*

Proof. The elements of the initial cluster $\mathcal{S}_n$ in $\mathcal{LM}_n$, which consists of the diametric pairs $\mathcal{D}_n$, are easily seen to be pairwise weakly separated. Then, by Proposition 6.3.5, any cluster we can reach from $\mathcal{S}_n$ using only mutations of the form (5.1.1) must also be pairwise weakly separated.

Conversely, consider any set C of $\binom{n}{2}$ pairwise weakly separated solid circular pairs. Let $C' = \{(P;Q)' \mid (P;Q) \in C\}$, and notice that $|C'| = 2\binom{n}{2}$. By Corollary 6.2.15 and Lemma 6.2.5, it is enough to prove that $C' \cup \{(\emptyset;\emptyset)\}$ is a solid cluster (see Definition 6.2.10) in $\mathcal{CM}_n$. From here it will follow by definition that C' is a symmetric solid seed, meaning C can be reached from $\mathcal{S}_n$ in $\mathcal{LM}_n$ using only mutations of the form (5.1.1), as desired.

Before proceeding, it is straightforward to check that circular pairs $(P;Q) = (p_1,\dots,p_a;q_1,\dots,q_a)$ and $(R;S) = (r_1,\dots,r_b;s_1,\dots,s_b)$ are weakly separated if and only if the following four intersections are non-empty:

$$[p_1,q_1] \cap (R \cup S), \quad [p_a,q_a] \cap (R \cup S), \quad [r_1,s_1] \cap (P \cup Q), \quad [r_b,s_b] \cap (P \cup Q),$$

where the interval $[a,b]$ denotes the vertices on the arc clockwise from a to b, inclusive. We now prove that $C' \cup \{(\emptyset;\emptyset)\}$ is a solid cluster. First, notice that if non-symmetric circular pairs $(P;Q)' = (p_1,\dots,p_a;q_1,\dots,q_a)'$ and $(R;S)' = (r_1,\dots,r_b;s_1,\dots,s_b)'$ are such that $k(P;Q)' = k(R;S)'$ and $T(P;Q)' = T(R;S)'$, but $D(P;Q)' \neq D(R;S)'$, then $(P;Q)$ and $(R;Q)$ are not weakly separated. Hence, at most one of $(P;Q)'$ and $(R;S)'$ is in C'. As there are exactly $2\binom{n}{2}$ choices of T and k that give valid non-symmetric solid circular pairs, there must be one element of C' corresponding to each choice of (T,k).

Second, consider any adjacent $(P;Q)'$ and $(R;S)'$ in C'. Without loss of generality, one of

- $k(P;Q)' = k(R;S)'$ and $T(P;Q)' = T(R;S)' + \frac{1}{2}$,
- $T(P;Q)' = T(R;S)'$ and $k(P;Q)' = k(R;S)' + 1$.

holds. In either case, because $(P;Q)'$ and $(R;S)'$ are weakly separated, we can see that $|D(P;Q)' - D(R;S)'| = 2$. It follows that $C' \cup \{(\emptyset;\emptyset)\}$ is a solid cluster, so we are done. $\square$

We now relate C and P. Recall that, by Lemma 6.2.6, if C satisfies C, then C is a positivity test. Furthermore, $|C| = \binom{n}{2}$. We can prove, similarly to [13, Theorem 1.2], that:

Proposition 6.3.7. *If C satisfies S, then $|C| \leq \binom{n}{2}$.*

In fact, we can prove a slightly stronger result by interpreting a circular pair as a set of edges.

Definition 6.3.8. For a circular pair $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$, define $E(P;Q) = \{\{p_i, q_i\} \mid i \in \{1, \ldots, k\}\}$. Similarly, for a set $D \subset \{\{i,j\} \mid 1 \leq i < j \leq n\}$ of edges such that no two edges in D cross, let $V(E)$ be the circular pair for which $E(V(D)) = D$.

Proposition 6.3.9. *If C is a set of pairwise weakly separated circular pairs with*

$$E = \bigcup_{(P;Q) \in C} E(P;Q),$$

then $|C| \leq |E|$.

Proof. Proceed by induction on $|E|$. The case $|E| = 0$ is trivial, so assume the result is true for $|E| < m$. Suppose that we have C, E with $|E| = m$, and assume for sake of contradiction that $|C| > m$. Choose some $\{a,b\} \in E$ such that, for any other $\{c,d\} \in E$, c and d do not both lie on the arc drawn from a to b in the clockwise direction (this arc is taken to include both a and b). Now, letting $E' = E \setminus \{\{a,b\}\}$, define the projection map $J : 2^E \to 2^{E'}$ by:

$$J(D) = \begin{cases} D \setminus \{\{a,b\}\} & \text{if } \{a,b\} \in D, \\ D & \text{otherwise.} \end{cases}$$

We may define J for circular pairs analogously: $J(P;Q) = V(J(E(P;Q)))$, and let $C' = \{J(P;Q) \mid (P;Q) \in C\}$. Then, the following two lemmas follow from straightforward casework:

Lemma 6.3.10. *The elements of C' are pairwise weakly separated.*

Lemma 6.3.11. *There is at most one $(P;Q) \in C$ with $\{a,b\} \in E(P;Q)$ and $J(P;Q) \in C$.*

Using these lemmas, we can finish our proof. By Lemma 6.3.10, the elements of C' are pairwise weakly separated, and we also have $E' = \bigcup_{(P;Q) \in C'} E(P;Q)$. Thus, by the inductive hypothesis, $|C'| \leq |E'| = |E| - 1$. However, we see from Lemma 6.3.11 that $|C'| \geq |C| - 1$, so the induction is complete. □

Now, Proposition 6.3.7 follows easily by taking $E = \{\{i,j\} \mid 1 \leq i < j \leq n\}$ in Proposition 6.3.9. Proposition 6.3.9 also has another natural corollary:

Corollary 6.3.12. *For any set S of pairwise weakly separated circular pairs, there is an injective map $e : S \to \{\{i,j\} \mid 1 \leq i < j \leq n\}$ such that $e(P;Q) \in E(P;Q)$ for each $(P;Q) \in S$.*

Proof. Proposition 6.3.9 gives exactly the condition required to apply Hall's marriage theorem. □

Acknowledgments

This work was undertaken at the REU (Research Experiences for Undergraduates) program at the University of Minnesota, Twin Cities, supported by NSF grants DMS-1067183 and DMS-1148634. The authors thank Joel Lewis and Pavlo Pylyavskyy for introducing them to this problem and for their invaluable insight and encouragement. The authors also thank Thomas McConville for helpful discussions, and Victor Reiner, Jonathan Schneider, and Dennis Stanton for suggesting references.

Finally, the authors thank the anonymous referees for comments and corrections.

Appendix A. Proofs of Lemmas 4.2.2–4.2.4

Proof of Lemma 4.2.2. We proceed by strong induction on n: the inequality is easily verified for $n = 6, 7, 8$ using Theorem 4.1.6. Furthermore, note that $X_n < 2nX_{n-1}$ for $n = 2, 3, 4, 5$ as well. Now, assume $n \geq 9$.

By Theorem 4.1.6, it is enough to show

$$X_{n-1} < \sum_{j=2}^{n-2}(j-1)X_jX_{n-j} < 2X_{n-1}. \tag{A.1}$$

We first show the left hand side of (A.1). Now, we have

$$\begin{aligned}
\sum_{j=2}^{n-2}(j-1)X_jX_{n-j} &> X_2X_{n-2} + (n-4)X_{n-3}X_3 + (n-3)X_{n-2}X_2 \\
&= 2(n-2)X_{n-2} + 8(n-4)X_{n-3} \\
&> \left(\frac{n-2}{n-1} + \frac{2(n-4)}{(n-2)(n-1)}\right)X_{n-1} \\
&> X_{n-1},
\end{aligned}$$

where we have applied the inductive hypothesis.

It remains to prove the right hand side of (A.1). First, suppose that n is odd, with $n = 2k-1, k \geq 4$. Let $Q_i = X_i/X_{i-1}$ for each i; we know that $Q_i > 2i-1$ for all $i \geq 5$. Then, we have

$$\begin{aligned}
\sum_{j=2}^{n-2}(j-1)X_jX_{n-j} &= (2k-3)\sum_{j=2}^{k-1}X_jX_{2k-1-j} \\
&= (2k-3)X_{n-1}\sum_{j=2}^{k-1}\frac{X_j}{Q_{2k-2}Q_{2k-3}\cdots Q_{2k-j}}
\end{aligned}$$

$$< (2k-3)X_{n-1}\sum_{j=2}^{k-1}\frac{X_j}{(4k-5)(4k-7)\cdots(4k-2j-1)}.$$

However, we claim that the terms in the sum are strictly decreasing. This amounts to the inequality $(4k-2j-1)X_{j-1} > X_j$ for $3 \le j \le k-1$, which follows by the inductive hypothesis as $4k-2j-1 > 2j$. Thus,

$$\begin{aligned}
&(2k-3)X_{n-1}\sum_{j=2}^{k-1}\frac{X_j}{(4k-5)(4k-7)\cdots(4k-2j-1)}\\
&\quad< (2k-3)X_{n-1}\left(\frac{X_2}{4k-5}+\frac{(k-3)X_3}{(4k-5)(4k-7)}\right)\\
&\quad= X_{n-1}\left(\frac{4k-6}{4k-5}+\frac{(4k-12)(4k-6)}{(4k-5)(4k-7)}\right)\\
&\quad< 2X_{n-1},
\end{aligned}$$

where we substitute $X_2 = 2, X_3 = 8$. The case in which n is even may be handled similarly, and the induction is complete. $\square$

Recall the definitions of D_n, E_n from Section 4.2. We now prove Lemmas 4.2.3 and 4.2.4, that $D_n/X_n \to \sqrt{e}-1$ and $E_n/X_n \to 0$, respectively.

Proof of Lemma 4.2.3. We may as well consider $D_n - 1 = \sum_{j=1}^{n-2}\binom{n}{j}X_{n-j}$. Using the notation $Q_i = X_i/X_{i-1}$, as in the proof of Lemma 4.2.2, we have

$$\begin{aligned}
\frac{\sum_{j=1}^{n-2}\binom{n}{j}X_{n-j}}{X_n} &= \sum_{j=1}^{n-2}\frac{1}{j!}\cdot\frac{n(n-1)\cdots(n-j+1)}{Q_nQ_{n-1}\cdots Q_{n-j+1}}\\
&= \sum_{j=1}^{n-2}\frac{1}{2^j j!}\cdot\frac{2n(2n-2)\cdots(2n-2j+2)}{Q_nQ_{n-1}\cdots Q_{n-j+1}}\\
&= \sum_{j=1}^{n-2}\frac{1}{2^j j!}+\sum_{j-1}^{n-2}\frac{1}{2^j j!}\left(\frac{2n(2n-2)\cdots(2n-2j+2)}{Q_nQ_{n-1}\cdots Q_{n-j+1}}-1\right).
\end{aligned}$$

As $n \to \infty$, first summand above converges to $\sqrt{e}-1$, so it is left to check that the second summand converges to zero.

Note that, by Lemma 4.2.2,

$$\frac{2n(2n-2)\cdots(2n-2j+1)}{Q_nQ_{n-1}\cdots Q_{n-j+2}} > 1.$$

Now,

$$\begin{aligned}0 &< \sum_{j=1}^{n-2} \frac{1}{2^j j!}\left(\frac{2n(2n-2)\cdots(2n-2j+2)}{Q_nQ_{n-1}\cdots Q_{n-j+1}} - 1\right) \\ &< \sum_{j=1}^{n-5} \frac{1}{2^j j!}\left(\frac{2n(2n-2)\cdots(2n-2j+2)}{Q_nQ_{n-1}\cdots Q_{n-j+1}} - 1\right) \\ &\quad + Kn\left[\frac{1}{2^{n-4}(n-4)!} + \frac{1}{2^{n-3}(n-3)!} + \frac{1}{2^{n-2}(n-2)!}\right],\end{aligned}$$

for some positive constant K.

It is also easy to see that the second term above goes to zero as $n \to \infty$. Now, applying Lemma 4.2.2 again (noting that the indices are all at least 6),

$$\begin{aligned}&\sum_{j=1}^{n-5} \frac{1}{2^j j!}\left(\frac{2n(2n-2)\cdots(2n-2j+2)}{Q_nQ_{n-1}\cdots Q_{n-j+1}} - 1\right) \\ &\quad < \sum_{j=1}^{n-5} \frac{1}{2^j j!} \cdot \left(\frac{2n}{2n-2j+1} - 1\right) \\ &\quad < \sum_{j=1}^{n-5} \frac{1}{2^{j-1}(j-1)!} \cdot \frac{1}{2n-2j+1}.\end{aligned}$$

It is not difficult to show that the above sum goes to zero as $n \to \infty$, so the proof is complete. □

Proof of Lemma 4.2.4. It is an immediate consequence (independently of the result of Theorem 4.2.1) of Lemma 4.2.2 that $X_n = O((2n-1)!!)$. Thus, to prove that $E_n/X_n \to 0$, we may as well prove that

$$n\sum_{j=2}^{n-2} \frac{(2j-1)!!(2n-2j-1)!!}{(2n-1)!!} \to 0.$$

It is straightforward to check that the largest terms of the sum are when $j = 2, n-2$, and these terms are of inverse quadratic order, from which the conclusion follows immediately. □

Appendix B. Proofs of Lemmas 5.2.5 and 5.2.6

We will first prove Lemma 5.2.5:

Lemma 5.2.5. *For any two incomplete circular pairs $(P;Q)$ and $(P';Q')$, any electrical positroid Z satisfying $S_0 \cup \{(P+1;Q+n)\} \subset Z \subset S_1$ must also contain $(P'+1;Q'+n)$.*

We start by presenting a series of technical lemmas.

Lemma B.1. *Let $(P;Q) = (a_1, \dots, a_k; b_1, \dots, b_k) \in S_0$ be an incomplete circular pair, such that $(P+1; Q+n)$ is a circular pair (and is not in S_0). Furthermore, assume that $(P;Q)$ is* ***minimal****, that is, $(P - a_k; Q - b_k)$ is complete. Then, for all $0 \le i \le k-1$, $(a_i; b_{i+1}), (a_{i+1}; b_i) \in S_0$.*

Proof. Immediate from axiom 1a of Definition 5.1.3. □

Lemma B.2. *Let $(a, b, c; d, e, f)$ be a circular pair. Then, if $(a; d), (a; f), (b; e), (c; d)$, $(c; f) \in S_0$, then $(a; d), (b; e), (b; f), (c; e) \in S_0$.*

Proof. Immediate from axiom 1b. □

Lemma B.3. *If $(a_1, \dots, a_n; b_1, \dots, b_n) \in S_0$, $(a_{n+1}; b_{n+1}) \in S_0$, and $a_n, a_{n+1}, b_{n+1}, b_n$ appear in clockwise order, then $(a_1, \dots, a_{n-1}, a_{n+1}; b_1, \dots, b_{n-1}, b_{n+1}) \in S_0$.*

Proof. If $(a_n; b_{n+1}) \in S$ and $(a_{n+1}; b_n) \in S$, the claim follows from axiom 2b, axiom 2c and induction on n. Otherwise, it follows from axiom 1a and induction on n. □

Lemma B.4. *Let $(P;Q) = (a_1, \dots, a_k; b_1, \dots, b_k) \in S_0$ be a complete circular pair. Then, $(P - a_i; Q - b_i)$ is complete for all $i = 1, 2, \dots, k$.*

Proof. Applying axiom 2c with a_1, a_i, a_k, b_1 to $(P; Q - b_k)$ gives $(P - a_i; Q - b_k) \in S$. Then another application of axiom 2c, to $(Q; P - a_i)$ with b_1, b_i, b_k, a_i gives the desired result. □

Lemma B.5. *Let $(P, a, b, c, Q; R, d, e, f, T)$ be a circular pair, where P, Q, R, T are sequences of boundary vertices. Suppose that*

$$(a; d), (a; e), (b; d), (b; e), (b; f), (c; e), (c; f) \in S,$$
$$(P, a, b; R, d, e) \in S, \quad \text{and}$$
$$(P, a, c, Q; R, d, f, T) \in S$$

Then, $(P, a, b, Q; R, d, e, T) \in S$.

Proof. It is straightforward to check that this lemma follows from induction on the size of P and axioms 2b and 2c. □

Lemma B.6. *Let P, Q, R, T be sequences of indices, and let $(1, P, a, b, c, Q; n, R, d, e, f, T)$ be a circular pair. Suppose*

$$(a;d),(a;e),(b;d),(b;e),(b;f),(c;e),(c;f) \in S,$$
$$(1,P,a,b,Q;n,R,d,e,T) \in S, \quad \textit{and}$$
$$(P,a,c,Q;R,d,f,T) \in S.$$

Then $(1,P,a,c,Q;n,R,d,f,T) \in S$.

Proof. It is straightforward to check that this lemma follows from induction and axiom 2c. □

Lemma B.7. *Consider a circular pair* $(P;Q) = (a_1,\dots,a_k;b_1,\dots,b_k)$*, and let* $(P+a;Q+b)$ *be an incomplete circular pair with* $a_k < a < b < b_k$ *in clockwise order. Then, any electrical positroid* Z *satisfying* $S_0 \cup \{(P+1;Q+n)\} \subset Z \subset S_1$ *contains* $(P+a+1;Q+b+n)$.

Proof. It is easy to see that any element of $Z\setminus S_0$ must be of the form $(P'+1;Q'+n)$, for some P',Q'. By axiom 1a, $(P+1;Q+n) \in Z$ and $(P+a;Q+b) \in Z$ implies that either $(P+a+1;Q+b+n) \in Z$, or $(P+a+1;Q+b+n) \notin Z$ and $(P+1;Q+b),(P+a,Q+2) \in Z$. We are done in the former case, so assume for sake of contradiction that we have the latter. $(P+1;Q+b),(P+a,Q+2)$ are not of the form $(P'+1;Q'+n)$, so cannot lie in $Z\setminus S$; thus, $(P+1;Q+b),(P+a,Q+2) \in S$. Finally, axiom 1b yields us $(P+1;Q+n) \in S$, a contradiction, so we are done. □

Definition B.8. Two pairs of indices (i,j) and (i',j') are said to **cross** if $i < i' < j' < j$ and $(i;j'),(i';j) \in S$.

Definition B.9. For ease of notation, denote the sequence of indices $a_k,\dots,a_\ell$ by $A_{k,\ell}$.

We now algorithmically construct a set $\mathcal{P}$ of circular pairs, which we will call **primary** circular pairs. The construction is as follows: begin by placing $(1;n) \in \mathcal{P}$. Then, for each $(P;Q) = (A_{1,i-1};B_{1,i-1}) \in \mathcal{P}$, if we also have $(P;Q) \in S_0$, perform the following operation.

- Let a be the first index appearing clockwise from a_{i-1} such that there exists c with (a,c) crossing (a_{i-1},b_{i-1}), and also $(A_{2,i-1},a;B_{2,i-1},c) \in S$. If a does not exist, stop. Otherwise, with a fixed, take c to be the first index appearing counterclockwise from b_{i-1} satisfying these properties.
- If a exists, add $(A_{1,i-1},a;B_{1,i-1},c)$ to $\mathcal{P}$, and remove $(P;Q) = (A_{1,i-1};B_{1,i-1})$.
- Similarly, let b to be the largest index counterclockwise from b_{i-1} such that there exists d with (d,b) crossing (a_{i-1},b_{i-1}) and $(a_2,\dots,d;b_2,\dots,b_i) \in S$. If b does not exist, stop. Otherwise, with b fixed, take d to be the first index clockwise from a_{i-1} with these properties.

- If $a \neq d$ and $b \neq c$ (note that if $a = d$, then $b = c$), then add $(A_{1,i-1}, d; B_{1,i-1}, b)$ to $\mathcal{P}$. Note that $c \leq d$ or else, by 1a, c could originally have been set to d.

It is easily seen that at any time, the algorithm may be performed on the elements of $\mathcal{P}$ in any order, and that it will eventually terminate, when the operation described above results in no change in $\mathcal{P}$ for all $(P; Q) \in \mathcal{P}$.

Definition B.10. For a circular pair $(P; Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$, define $E(P; Q) = \{\{p_i, q_i\} \mid i \in \{1, \ldots, k\}\}$. We will take $E(P; Q)$ to be an ordered set and abusively refer to its elements as **connections**.

Lemma B.11. *For any incomplete circular pair $(P; Q)$, there exists a circular pair $(P'; Q') \in \mathcal{P}$ such that any electrical positroid Z satisfying $S_0 \cup \{(P'; Q')\} \subset Z \subset S_1$ contains $(P + 1; Q + n)$.*

Proof. By Lemma B.7, we may assume that $(P; Q)$ is a minimal incomplete circular pair. Consider the primary circular pairs whose first i connections are the same as those of $(P; Q)$. By the construction of $\mathcal{P}$, there are at most two such primary circular pairs, and we can choose either.

Then, we can prove the lemma by retrograde induction on i, where i is such that the first i connections of $(P; Q)$ are shared with some primary circular pair. As this induction is straightforward from the previous lemmas, we omit the details. □

Lemma B.12. *There is exactly one circular pair in $\mathcal{P}$ that does not lie in S_0, which we call the S_0-**primary circular pair**.*

Proof. By Lemma B.11, $\mathcal{P} \setminus S_0$ has at least one element, because S_0 does not have the $(n, 1)$-BEP. It is then straightforward to check that the axioms do not allow us to have more than one element, so we are done. □

Lemma B.13. *For any incomplete circular pair $(P; Q)$, any electrical positroid Z satisfying $S_0 \cup \{(P + 1; Q + n)\} \subset Z \subset S_1$ contains the S_0-primary circular pair.*

Proof. The argument is a retrograde induction on i, where i is such that the first i connections of $(P; Q)$ are the same as those of some primary circular pair. The base case is immediate from the subset axiom, and the induction follows from applications of the axioms and Lemma B.5. □

We then have our desired lemma as an immediate corollary of Lemmas B.13 and B.11. Next, we prove Lemma 5.2.6:

Lemma 5.2.6. *There exists an electrical positroid T with $T_0 \subset T \subset S_0 \cap S_0'$.*

Proof. We give an algorithm to construct such an electrical positroid T. We begin by setting $T = T_0$; note that T satisfies the last two electrical positroid axioms, but may not satisfy the first six. Each of the first six axioms are of the form $\mathcal{A}, \mathcal{B} \in T \Rightarrow \mathcal{C}, \mathcal{D} \in T$, or otherwise $\mathcal{A}, \mathcal{B} \in T \Rightarrow \mathcal{C}, \mathcal{D} \in T$ or $\mathcal{E}, \mathcal{F} \in T$. At each step of the algorithm, if T is an electrical positroid, we stop, and if not, we pick an electrical positroid axiom α (among the first six) not satisfied by $\mathcal{A}, \mathcal{B} \in T$. We then show that we can add elements of $S_0 \cap S_0'$ to T so that α is satisfied by $\mathcal{A}, \mathcal{B}$, and so that T also still satisfies the subset axiom.

It is clear what to do when α is one of axioms 1b, 1c, 2b, and 2c: we simply add the circular pairs $\mathcal{C}, \mathcal{D}$, and their subsets. Thus, we need to check what to do when α is one of axioms 1a and 2a. Then, it will be clear that the algorithm must terminate, because we can only add finitely many elements, with T having the desired properties.

We first consider axiom 1a, which we assume to fail in T when applied to $(P - a; Q - c), (P - b; Q - d) \in T$. If $(P - a; Q - c), (P - b; Q - d) \in S_0$, either $(P - a; Q - d), (P - b; Q - c) \in S_0$ or $(P - a - b; Q - c - d), (P; Q) \in S_0$. It is easy to see that $1 \in P$ and $n \in Q$ (or vice versa, but we can swap P and Q and reverse their orders), or else axiom 1a already would have been satisfied by $(P - a; Q - c), (P - b; Q - d) \in T$. Then, we just need to perform a straightforward casework check based on whether a and c are equal to 1 or n.

Axiom 2a can be handled in a similar manner, by casework on whether a and d are 1 or n, and so our algorithm is well-defined. $\square$

References

[1] A. Björner, et al., Oriented Matroids, Encyclopedia Math. Appl., Cambridge University Press, 1999.

[2] A. Björner, M. Wachs, Bruhat order of Coxeter groups and shellability, Adv. Math. 43 (1982) 87–100.

[3] D. Callan, Counting stabilized-interval-free permutations, J. Integer Seq. 7 (2004) 04.1.8.

[4] E.B. Curtis, D. Ingerman, J.A. Morrow, Circular planar graphs and resistor networks, Linear Algebra Appl. 283 (1998) 115–150.

[5] Y.C. de Verdière, I. Gitler, D. Vertigan, Réseaux électriques planaires II, Comment. Math. Helv. 71 (1996) 144–167.

[6] G. Dupont, F. Palesi, Quasi-cluster algebras associated to non-orientable surfaces, arXiv:1105.1560v1 [math.RA], 2011.

[7] S. Fomin, P. Pylyavskyy, Tensor diagrams and cluster algebras, arXiv:1210.1888 [math.CO], 2012.

[8] S. Fomin, A. Zelevinsky, Total positivity: tests and parametrizations, Math. Intelligencer 22 (2000) 23–33.

[9] S. Fomin, A. Zelevinsky, Cluster algebras, J. Amer. Math. Soc. 15 (2002) 497–529.

[10] R. Kenyon, The Laplacian on planar graphs and graphs on surfaces, arXiv:1203.1256 [math.PR], 2012.

[11] A. Knutson, T. Lam, D. Speyer, Positroid varieties: juggling and geometry, arXiv:1111.3660 [math.AG], 2011.

[12] T. Lam, P. Pylyavskyy, Laurent phenomenon algebras, arXiv:1206.2611 [math.RT], 2012.

[13] B. Leclerc, A. Zelevinsky, Quasicommuting families of quantum Plücker coordinates, Amer. Math. Soc. Transl. Ser. 2 181 (1998).

[14] OEIS Foundation Inc., 2011, The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A111088.

[15] S. Oh, A. Postnikov, D. Speyer, Weak separation and plabic graphs, arXiv:1109.4434 [math.CO], 2011.

[16] A. Postnikov, Total positivity, Grassmannians, and networks, arXiv:math.CO/0609764, 2006.

[17] J. Riordan, The distribution of crossings of chords joining pairs of $2n$ points on a circle, Math. Comp. 29 (1975) 215–222.
[18] P. Salvatore, R. Tauroso, The operad Lie is free, J. Pure Appl. Algebra 213 (2009) 224–230.
[19] J. Scott, Quasi-commuting families of quantum minors, J. Algebra 290 (2005) 204–220.
[20] W.A. Stein, et al., Sage Mathematics Software (Version 5.9), The Sage Development Team, 2013.
[21] P.J. Stein, C.J. Everett, On a class of linked diagrams II. Asymptotics, Discrete Math. 21 (1978) 309–318.